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Closed extensions of R-modules in the case of a semi-artinian ring R

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Algebra. — *Closed extensions of* R–*modules in the case of a semi–artinian ring* R. Nota (*) di FRANS LOONSTRA, presentata dal Socio G. ZAPPA.

RIASSUNTO. — Si considerano le estensioni chiuse B di un R-modulo A mediante un R-modulo C nel caso in cui R sia un anello semi-artiniano, cioè un anello R con la proprietà che per ogni quoziente $(R/_I \neq 0$ sia soc $(R/I) \neq 0$. Tali estensioni sono caratterizzate dal fatto che A deve essere un sottomodulo semi-puro di B.

§ 1. INTRODUCTION

In connection with the definition of an essentially closed submodule we pay attention to the closed extensions of R-modules, in particular (§ 3) in the case of a semi-artinian ring R, i.e. of a ring R with the property that for every non-zero quotient R/I with respect to a left ideal I of R, soc $(R/I) \neq 0$. In § 2 we start with some important properties of R-modules over semi-artinian rings; in § 3 the closed submodules and the closed extensions of these R-modules are characterized.

The exact sequence

$$E : 0 \to A \xrightarrow{\mu} B \xrightarrow{\nu} C \to 0 \tag{1}$$

of left R-modules is called a *closed extension* of A by C if $\mu(A)$ is an (essentially) closed submodules of B (notation: $\mu(A) \subseteq_{cl} B$). If E, E′ are congruent extensions of A by C (in the usual sense) then E′ is a closed extension if E is closed. We denote by $\mathrm{Ext}_R^{cl}(C ; A)$ the set of all congruence classes of closed extensions of A by C.

Following the usual procedure we can construct the induced extensions of (1) by means of an R-homomorphism $\gamma : C' \to C$, resp. $\alpha : A \to A'$. It can be proved that if (1) is a closed extension of A by C, then the induced extension $E' = E\gamma$ (resp. $E'' = \alpha E$) is a closed extension of A by C′ (resp. of A′ by C); see Loonstra [1]. For the closed extensions E, E′ of A by C the sum

$$E + E' = \nabla_A (E \oplus E') \Delta_C \tag{2}$$

(*) Pervenuta all'Accademia il 30 luglio 1985.

(1) F. LOONSTRA, *Essential submodules and essential subdirect products*, «Symp Math», vol. 23, 1979, Rome; 85-105.

is again a closed extension of A by C, and the set Ext^{cl}_R (C ; A) of all congruence classes of closed extensions of A by C is an abelian group under the binary operation which assigns to the congruence class of the extensions E and E′ the congruence class of the extension E + E′ given in (2).

The class of the splitting extension $0 \to A \to A \oplus C \to C \to 0$ is the zero element of this group, while the inverse of a closed extension E is the (closed) extension (-1_A) E. The exact sequences on Hom_R and Ext^{cl}_R can be amalgamated into long exact sequences (see [1]). We are now interested in the character of the closed extensions in the particular situation that R is a semi-artinian ring.

§ 2. Semi-artinian modules and rings

We start with the definition of a semi-artinian R-module ${}_R M$:

2.1. Definition: *An* R-*module* ${}_R M$ *is semi-artinian if and only if for every non-zero quotient* M/N, soc (M/N) $\neq 0$.

Then we have the following equivalent properties:

2.2. (*a*) ${}_R R$ *is a semi-artinian ring*;

(*b*) *every* R-*module is semi-artinian*;

(*c*) *every* R-*module* ($\neq 0$) *has a socle* $\neq 0$;

(*d*) *every* R-*module* ($\neq 0$) *is an essential extension of its socle.*

For the proofs see e.g. *Stenström*, Rings of quotients Ch. VIII, § 2.

2.3. *The* R-*module* ${}_R M$ *is semi-artinian if and only if* M *belongs to the hereditary torsion class generated by the class* C *of all simple* R-*modules.*

Indeed, the torsion class $\mathscr{I}_C$ generated by all simple modules consists of all modules M such that each non-zero quotient of M has a non-zero submodule of C, i.e. contains a simple submodule, i.e. $\mathscr{I}_C$ consists of semi-artinian R-modules.

2.4. *If* R *is a semi-artinian ring without zero-divisors, then* R *is a division ring.*

Proof: If L is a minimal left ideal of R, then $L^2 \subseteq L$ implies that $L^2 = L$; therefore there exists an idempotent $e \in R$ such that $L = Re$. Since $e(1-e) = 0$, e is a zero-divisor of R.

Since R is a domain, we have $e = 1$ and $L = R$, i.e. R is a division ring.

A category of R-modules is called a *locally-uniform* category (= l.u.-category) if every R-module $M \neq 0$ contains a non-zero uniform submodule. We say: R is an *l.u.-ring* if the category R-Mod is an l.u.-category. Since every simple R-module is uniform, this implies that a semi-artinian ring R is a locally uniform ring. That brings us to the result:

2.5. *If* R *is a semi-artinian ring,* $M \neq 0$ *an* R*-module, then there exists a maximal independent set of non-zero uniform submodules* U_i $(i \in I)$ *of* M. *Choosing suitable complements* U_i^c $(i \in I)$ *of the* U_i, *we have*:

(*a*) $S = \sum_{i \in I} U_i = \oplus_i U_i \subseteq_e M$;

(*b*) $\bigcap_{i \in I} U_i^c = 0$ *is an irredundant intersection of maximal* (*essentially*) *closed submodules* U_i^c *of* M; *the* U_i^c *are moreover irreducible submodules of* M;

(*c*) M *can be represented as a subdirect product* $M \overset{\times}{\underset{i \in I}{\simeq}} M/U_i^c$ *of uniform modules* M/U_i^c .

Proof: (*a*) If V is a submodule of M, $S \cap V = 0$, then V contains a uniform submodule $\neq 0$, contradicting the maximality of $\{U_i \mid i \in I\}$.

(*b*) Let $U_{i_0}^c \supsetneqq \oplus_{i \neq i_0} U_i$ be a complement of U_i; then $\bigcap_{i \neq i_0} U_i^c \supseteq U_{i_0}$ $(\forall i_0)$, and $\bigcap_{i \in I} U_i^c = 0$; so $\bigcap_{i \in I} U_i^c = 0$ is an irredundant intersection of maximal (essentially) closed submodules U_i^c $(i \in I)$ of M. From $\pi_i(U_i) \cong U_i \subseteq_e M/U_i^c$ (where π_i is the canonical projection $M \rightarrow M/U_i^c$) it follows that the M/U_i^c are uniform and that the U_i^c are irreducible.

2.6. *Let* R *be a semi-artinian ring*; *then the following conditions are equivalent*:

(i) N *is an essentially closed submodule of* M;

(ii) *If* I *is a maximal left ideal of* R, $m \in M$, *and* $Im \subseteq N$, *then* $\exists\, n \in N$, *such that* $i \cdot m = i \cdot n$ $(\forall i \in I)$.

Proof: (i) $\rightarrow$ (ii) Assume that $m \notin N$; then $(Rm + N)/N$ is a minimal submodule of M/N. If N^c is a complement of N (in M), then $(N \oplus N^c)/N \subseteq$ $\subseteq_e M/N$, and since $(Rm + N)/N$ is minimal in M/N, we have $(Rm + N)/N \subseteq$ $(N \oplus N^c)/N$, i.e. $Rm \subseteq N \oplus N^c$, or $m \in N \oplus N^c$, i.e. $m = n + n'$, for some $n \in N$, $n' \in N^c$. Since $Im \subseteq N$, we have $In' = 0$, or $I(m - n) = 0$, or $i \cdot m =$ $= i \cdot n$ $(\forall i \in I)$.

(ii) $\rightarrow$ (i) We have to prove that $N^{cc} = N$; suppose that $N^{cc} \supsetneqq N$. Choose $n'' \in N^{cc}$, $n'' \notin N$.

Since $N \subset_e N^{cc}$, there exists $0 \neq r_0 \in R$ with $0 \neq r_0 n'' \in N$. Mapping $r \mapsto rn'' + N$, we find $R/(N : n'') \cong (Rn'' + N)/N \subseteq N^{cc}/N$. Since R is semi-artinian, every R-module is semi-artinian, i.e. N^{cc}/N contains a minimal submodule $(Rm + N)/N$, $m \notin N$. Since $L = (N : m)$ is a maximal left ideal of R, and $Lm \subseteq N$, there exists (by assumption) an element $n \in N$ such that $L(m - n) = 0$.

Then R $(m - n)$ is a minimal submodule of N^{cc}, therefore R $(m - n) \subseteq$ N, since N $\subseteq_e N^{cc}$.

But $m - n \in$ N and $n \in$ N leads to a contradiction $m \in$ N! Therefore $N^{cc} =$ N.

We apply 2.6. in the following result.

2.7. *Let* M *be a semi-artinian* R-*module*; *then* M *is injective if every* R-*homomorphism of a maximal left ideal* L *of* R *into* M *can be extended to an* R-*homomorphism of* R *into* M.

Proof. The necessity of the condition is obvious. Conversely: let $m' \in$ M, then for $(M : m') = \{r \in R \mid rm' \in M\} = L'$ we have $L'm' \subseteq$ M. Let L be a maximal left ideal containing L′, then $Lm' \subseteq$ M. Define $\varphi : L \to M$ by $\varphi : r \mapsto rm'$, then φ is an R-homomorphism of L into M, and by assumption φ can be extended to an R-hom. $\varnothing : R \to M$. Let $\varnothing(1) = m$, then for $r \in L$ we have $\varphi(r) = \varnothing(r)$, i.e. $rm' = rm$, $r(m' - m) = 0$ $(\forall r \in L)$. By 2.6. it follows that M is an essentially closed submodule of $\hat{M}$, i.e. M is injective, $M = \hat{M}$.

2.8. *Let the semi-artinian ring* R *be a left principal ideal ring*; *then we have*:

(i) N *is an ess*^{y} *closed submodule of* M *iff* $N \cap pM = pN$ *for each element* $p \in$ R *generating a maximal left ideal of* R.

(ii) *If* $pM = M$ *for each element* $p \in$ R *generating a maximal left ideal of* R, *then* M *is injective.*

Proof. (i) If Rp is an arbitrary maximal left ideal of R, then the condition: $(Rp)\, m \subseteq N$ $(m \in M)$ implies $Rp\,(m - n) = 0$ for some $n \in$ N and that is equivalent to $N \cap pM = pN$. Hence (i) is a consequence of 2.6 (ii).

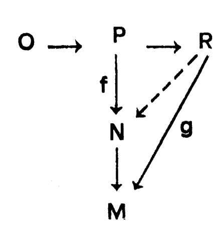
(ii) If $\hat{M}$ is an injective hull of M, and $M \cup p\hat{M} \subseteq M = pM$, it follows from (i) that M is a closed submodule of the injective module $\hat{M}$; i.e. $M = \hat{M}$.

Note. The ring Z is a semi-artinian principal ideal ring. In this case we find (again) that —exactly—the *neat* subgroups of an abelian group G are the essentially closed subgroups of G. If moreover $pG = G$ for every prime number $p \in$ Z, then we find the well-known result, that G is a divisible group.

§ 3. Semi-pure submodules of semi-artinian R-modules

3.1. Definition. *Let* N *be a submodule of the* R-*module* M (*here* R *is any ring*!); *then* N *is called a semi-pure submodule of* M *if* N *satisfies the following condition*:

Let P *be a maximal left ideal of* R, $f : P \to N$ *an* R-*homom.having an extension* $g : R \to M$; *then* f *has an extension of* R *into* N.



3.2. *An essentially closed submodule* N *of any* R-*module* M *is semi-pure.*

Proof. Let P be a maximal left ideal of R, $f : P \to N$ an R-homom.having an extension $g : R \to M$. Then $g(r) = rm$ ($\forall r \in R$) for some $m \in M$. Let $\hat{N}$ be an injective hull of N in $\hat{M}$, then $N = \hat{N} \cap M$, and there exists a homomorphism $f' : R \to \hat{N}$ extending f; i.e. we have $f'(r) = rm_1$ ($\forall r \in R$) for some $m_1 \in \hat{N}$. If $m = m_1$, then $m \in M \cap \hat{N} = N$ —and in that case—the proof is given. If $m_1 \neq m$, then $pm = pm_1$ ($\forall p \in P$), i.e. $\mathrm{Ann}_R(m - m_1) = P$, i.e. $m - m_1 \in \mathrm{soc}(\hat{M})$, and since $\mathrm{soc}(\hat{M})$ is the intersection of all essential submodules of $\hat{M}$, and $M \subseteq_e \hat{M}$, we have $m - m_1 \in M$, i.e. $m_1 \in M$. Then $m_1 \in M \cap \hat{N} = N$.

3.3. *Let* R *be a semi-artinian ring and* M *an* R-*module*; *then the following properties of a submodule* N *of* M *are equivalent:*

(i) N *is an essentially closed submodule of* M.

(ii) N *is a semi-pure submodule of* M.

Proof. (i) $\to$ (ii) has been proved in 3.2. (ii) $\to$ (i) Let N be a semi-pure submodule of M, and N′ an essential extension of N in M. If $N' \neq N$, then $\mathrm{soc}(N'/N) \neq 0$, and there exists a maximal left ideal P of R, and $n' \in N'$, $n' \notin N$, such that $Pn' \subseteq N$. Let $f : P \to N$ be defined by $f(p) = pn'$ ($\forall p \in P$), then f has an extension $f' : R \to M$ defined by $f'(r) = rn'$. Then (just as in 3.2) we can find $n \in N$ such that $f(p) = pn$ ($\forall p \in P$) and $Ann_R(n - n') = P$, i.e. $n - n' \in \mathrm{soc}(N')$, therefore $n - n' \in \mathrm{soc}(N) = \mathrm{soc}(N')$. That implies that $n' \in N$, i.e. $N' = N$ and so N is essentially closed.

Comparing the results of 2.6 and 3.3 we find:

COROLLARY 3.4. *Let* R *be a semi-artinian ring*; *then the following properties of an* (*essentially*) *closed submodule* N *of the* R-*module* M *are equivalent*:

(i) N *is a semi-pure submodule of* M;

(ii) *Let* I *be a maximal left ideal of* R, $m \in M$ *and* $Im \subseteq N$, *then there exists an element* $n \in N$, *such that* $im = in$ ($\forall i \in I$).

Let us now pay attention to the case that R is a *semi-artinian* ring, i.e. that for every non-zero quotient R/I we have $\mathrm{soc}(R/I) \neq 0$. We have to characterize the *closed extensions* E of A by C. We remember that A is closed in B if and only if A is a semi-pure submodule of B; i.e. that if P is a maximal left ideal of R, $b \in B$ and $Pb \subseteq A$, then there is an element $a \in A$ such that $pa = pb$ ($\forall p \in P$). The last condition can be stated as follows: if $Pb \subseteq A$ for some $b \in B$, then $pb = a_0$ impies that there is an $a \in A$ such that

$$pa = pb = a_0;$$

or: the equation $px = a_0$ ($p \in P$) is solvable in A whenever it is solvable in B. This is equivalent to the requirement

(3) $$pA = A \cap pB \ (\forall p \in P, \ P \text{ being a max. left ideal of } R).$$

3.5. *The closed extensions* B *of* A *by* C *are therefore characterized by the fact that* A *must be a semi-pure subgroup of* B *and this condition can be given by* (3).

Note that Z is a semi-artinian ring; in that case the closed subgroups A of an abelian group are the *neat* subgroups, and the closed extensions E of A by C are the neat extensions of A by C; see Schoeman [2].

(2) M.J. SCHOEMAN, *The group of neat extensions*, Doct. thesis, 1970, Delft.