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On the ampleness of $K_X \otimes L^n$ for a polarized threefold $(X, L)$


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Geometria algebrica. — On the ampleness of \( K_X \otimes L^n \) for a polarized threefold \((X, L)\) \((*)\). Nota di Antonio Lanteri e Marino Palleschi \((**)\), presentata \((***)\) dal Socio E. Marchionna.

RIASSUNTO. — Siano X una varietà algebrica proiettiva complessa non singolare tridimensionale, L un fibrato lineare ampio su X, e \( n \geq 2 \) un intero. Si prova che, a meno di contrarre un numero finito di \((-1\))-piani di X, il fibrato \( K_X \otimes L^n \) è ampio ad eccezione di alcuni casi esplicitamente descritti. Come applicazione si dimostra l'ampiezza del divisore di ramificazione di un qualunque rivestimento di \( P^3 \) o della quadrica liscia di \( P^4 \).

1. INTRODUCTION

The subject of this paper is the ampleness of \( K_X \otimes L^n \) on a polarized threefold, i.e. a pair \((X, L)\) where X is a complex connected projective algebraic threefold and L an ample line bundle on X. We prove that, up to contracting a finite number of \((-1\))-planes of X, \( K_X \otimes L^n \) is ample if \( n \geq 2 \), apart from a few cases explicitly described (Theorem 2.1). This fact together with known results on surfaces [5] implies that \( K_X \otimes L^{k+1} \) is ample for any polarized manifold \((X, L) \neq (P^k, \mathcal{O}_{P^k}(1))\) of dimension \( k \leq 3 \). If the same were true in every dimension, it would extend Ein's result on the ampleness of the ramification divisor of a branched covering of \( P^k \) [2]. Partial results are provided by Propositions 2.4, 2.5. On the other hand Ein's result can be generalized in a different perspective. Actually, as an application of Theorem 2.1, we show the ampleness of the ramification divisor of any branched covering of a quadric threefold (Theorem 3.2).

2.

Let \((X, L)\) be a polarized threefold. As usual we shall not distinguish between line bundles and invertible sheaves. We also write \( L^n \) for \( L^{\otimes n} \). Following Sommese [6] we shall call a polarized threefold \((X', L')\) a reduction of \((X, L)\) if there is a surjective morphism \( \pi : X \to X' \) such that i) \( \pi \) is the blow-up of a finite set \( F \subset X' \) and ii) \( \pi^* L' = L \otimes [\pi^{-1}(F)] \). \( K_X \) will stand for the canonical bundle of X.

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(2.1) **THEOREM.** Let \((X, L)\) be a polarized threefold. The line bundle \(K_X \otimes L^n\) is ample for \(n \geq 2\), apart from the following cases:

- **\(n = 4\)** and \((X, L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))\);
- **\(n = 3\)** and either \((X, L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)), (X, L) = (Q, \mathcal{O}_Q(1)), Q\) being a smooth hyperquadric of \(\mathbb{P}^4\), or \(X\) is a \(\mathbb{P}^2\)-bundle and \(L_1/F = \mathcal{O}_{\mathbb{P}^2}(1)\) for any fibre \(F\) of \(X\);
- **\(n = 2\)** and either
  - (a) \(X\) is a \(\mathbb{P}^1\)-bundle and \(L_1/F = \mathcal{O}_{\mathbb{P}^1}(1)\),
  - (b) \(X\) is a \(\mathbb{P}^2\)-bundle and \(L_1/F = \mathcal{O}_{\mathbb{P}^2}(e), e = 1\) or 2,
  - (c) \(X\) is a quadric bundle and \(L_1/F = \mathcal{O}_{\mathbb{P}^2}(1)\), where, in each case, \(F\) is a fibre of \(X\),
  - (d) \(X\) is a Fano threefold of index \(r \geq 2\), \(\text{Pic}(X) \simeq \mathbb{Z}[l]\) and \(L = l^m, m < r\), or there is a reduction \((X', L')\) of \((X, L)\) where \(K_{X'}, \otimes L'^2\) is ample.

**Proof.** Let us consider the line bundle \(N = K_X \otimes L^n\). First assume that \(N^n\) is spanned by its global sections for some \(s > 0\). So by tensoring \(N^n\) with the ample line bundle \(L^s\), we get the ampleness of \(N \otimes L = K_X \otimes L^n\).

Now assume that for no \(s > 0\) \(N^n\) is spanned by its global sections and let \(M = N \otimes K_X^{-1} = L^n^{-1}\). Then since \(K_X^n \otimes M^n\) is spanned for no \(n > 0\), it follows from [1, Thm. 2.2] that either \((X, M)\) is one of the pairs listed in (a)-(d) or it admits a reduction \((X', M')\) such that some power of \(K_{X'} \otimes M'\) is spanned by its global sections. In the latter case, let \(\pi : X \to X'\) be the reduction morphism and let \(E_1, \ldots, E_t\) be the \((-1)\)-planes contracted by \(\pi\). Since \(\pi^* M' = M \otimes [E_1] \otimes \cdots \otimes [E_t]\), by restriction to \(E_0\), we get

\[M_{|E_0} = [E_i]^{-1} \otimes E_i = \mathcal{O}_{\mathbb{P}^2}(1).\]

On the other hand \(M = L^n^{-1}\) and therefore this case can occur only when \(n = 2\). It only remains to see which of the exceptions (a)-(d) are allowable for the pair \((X, M)\) when \(n \geq 3\). Since \(M = L^n^{-1}\), cases (a) and (c) cannot occur, whereas case (b) happens if and only if \(M_1/F = \mathcal{O}_{\mathbb{P}^2}(2)\), which means that \(n = 3\) and \((X, L)\) is as in (b) with \(e = 1\). Finally assume that \((X, M)\) is as in (d). Then we have

\[L^n^{-1} = M = l^m, m < r,\]

where \(l\) is the ample generator of \(\text{Pic}(X)\) and \(r\) is the index of the Fano threefold \(X\). Since \(r \leq 4\) and \(n \geq 3\) we have the following possibilities: \(n = 4 = r\), in which case \(X \simeq \mathbb{P}^3\) and \(L = l\), \(n = 3 \leq r \leq 4\), in which case \(L = l\) and \(X\) is either \(\mathbb{P}^3\) or a quadric hypersurface. **q. e. d.**

By summing up some known results in dimension less than 3, we get
(2.2) COROLLARY. Let \((X , L)\) be a polarized manifold of dimension \(k \leq 3\). If \((X , L) \cong (\mathbb{P}^k , \mathcal{O}_{\mathbb{P}^k}(1))\), then \(K_X \otimes L^n\) is ample for any \(n \geq k + 1\).

For \(k = 1\) this is a trivial fact; for \(k = 2\) see [5].

This suggests the following

(2.3) QUESTION. Is \(K_X \otimes L^{k+1}\) ample for any dimension \(k \geq 1\), apart from the obvious exception \((\mathbb{P}^k , \mathcal{O}_{\mathbb{P}^k}(1))\)?

As is known the answer is affirmative when \(L\) is very ample. This could be deduced indirectly from the finiteness of the Gauss map [3] and Lemma 4 of [2]; but, what's more, using the standard technique of separating points and tangent vectors, one can directly prove, by induction.

(2.4) PROPOSITION. If \(L\) is very ample, then \(K_X \otimes L^{k+1}\) is very ample unless \((X , L) \cong (\mathbb{P}^k , \mathcal{O}_{\mathbb{P}^k}(1))\).

In a very special case (2.3) can be answered affirmatively.

(2.5) PROPOSITION. Assume that \(\text{Pic} (X) \cong \mathbb{Z}\); then \(K_X \otimes L^{k+1}\) is ample unless \((X , L) \cong (\mathbb{P}^k , \mathcal{O}_{\mathbb{P}^k}(1))\).

\textbf{Proof.} Let \(l\) be the ample generator of \(\text{Pic} (X)\). Then \(K_X = l^r\), \(r \in \mathbb{Z}\). Of course there is nothing to prove when \(r \leq 0\) and so we can assume \(r > 0\). This means that \(X\) is a Fano manifold of index \(r\). Let \(L = l^n\), \(n > 0\) and assume that \(K_X \otimes L^{k+1} = l^{n(k+1) - r}\) is not ample. This yields \(n(k + 1) - r < 0\). If equality occurs, then we get \((X , L) \cong (\mathbb{P}^k , \mathcal{O}_{\mathbb{P}^k}(1))\), by [4], Th. 2.1. So it is enough to prove that it cannot be \(r > n(k + 1)\). Actually we prove that \(r \leq k + 1\). To see this, put \(\chi(m) = \chi(l^m)\). By the Kodaira vanishing theorem we know that \(h^i(l^m) = 0\) if \(m < 0\) and \(i = 0, \ldots , k\). So, by Serre's duality, \(\chi(m) = (-)^k h^k(l^m) = (-)^k h^0(l^{-1}m)\) if \(m < 0\). Therefore

\[(2.5.1) \quad \chi(m) = 0 \quad \text{for} \quad -r \leq m < 0.\]

On the other hand, since \(X\) is Fano, \(\chi(0) = \chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) = 1\) by the Kodaira vanishing theorem again. Hence \(\chi(m)\), a polynomial of degree \(k\) which does not vanish everywhere, has at most \(k + 1\) distinct roots. It thus follows from (2.5.1) that \(r \leq k + 1\). \(\text{q. e. d.}\)

3.

Let \(f : X \to Y\) be a finite morphism of projective manifolds of dimension \(k\). The ramification formula gives \(R \in |K_X \otimes f^* K_Y^t|\), where \(R\) stands for the ramification divisor of \(f\) on \(X\). Now assume that \(K_Y^t = N^t\), with \(N\) ample and \(t > 0\) (this is equivalent to saying that \(Y\) is a Fano manifold whose index
Since $f$ is a finite morphism, the line bundle $L = f^* N$ is ample and

$$(3.0) \quad R \in |K_X \otimes L^t|, \text{ } L \text{ ample.}$$

Hence (2.1) applies to studying the ampleness of the ramification divisor of branched coverings of Fano manifolds.

(3.1) Example. Take $Y = P^k$; so (3.0) becomes $R \in |K_X \otimes L^k+1|$. By Corollary 2.2, if $k \leq 3$ we get the ampleness of $R$ with the trivial exception $\deg f = 1$. If the answer to Question 2.3 were affirmative, then we could obtain the ampleness of $R$ for any $k$. Actually the ampleness of the ramification divisor of a branched covering of $P^k$ was proved by Ein [2] answering a question asked by Lazarsfeld. This fact might be a good reason to hope that the answer to (2.3) is yes.

At least when $k \leq 3$, the above results allow us to study the ampleness of $R$ when $P^k$ (i.e. the Fano manifold of index $r = k + 1$) is replaced with a quadric (i.e. a Fano manifold of index $r = k$).

(3.2) Theorem. Let $f : X \to Q$ be a finite morphism from a manifold $X$ to a smooth quadric $Q$ of dimension $k \leq 3$. The ramification divisor $R$ is ample unless either

i) $f$ is an isomorphism, or 

ii) $k = 2$, $X$ is a $P^1$-bundle, $f^* \mathcal{O}_Q(1)_{|F} = \mathcal{O}_{P^1}(1)$ for any fibre $F$ of $X$ and $R$ is a sum of fibres.

Proof. Using the above notation, $R \in |K_X \otimes f^* \mathcal{O}_Q(k)|$, since $K_Q = \mathcal{O}_Q(1-k)$. Then $R$ is ample unless either

a) $(X,L) \simeq (P^k, \mathcal{O}_{P^k}(1)),$ 

b) $(X,L) \simeq (Q, \mathcal{O}_Q(1)),$ or 

c) $X$ is a $P^{k-1}$-bundle and $L_{|F} = \mathcal{O}_{P^{k-1}}(1)$ for any fibre $F$ of $X$.

This follows from Theorem 2.1 for $k = 3$ and from [5], Th. 2.5, in case $k = 2$.

Let $H \in |\mathcal{O}_Q(1)|$; then

$$(f^* H)^k = (\deg f)(H)^k = 2 \deg f.$$ 

But $(f^* H)^k = 1$ or $2$ according to whether we are in case (a) or (b). Therefore case (a) cannot occur, whereas $\deg f = 1$ in case (b). In case (c) let $g = f_{|F}$; since $g^* \mathcal{O}_Q(1) = \mathcal{O}_{P^{k-1}}(1)$, $g$ embeds $F (= P^{k-1})$ into $Q$ as a linear space of dimension $k - 1$. As $Q$ is assumed to be smooth, this can only occur when $k = 2$. In this case $K_X|_F = \mathcal{O}_{P^1}(-2)$, since $X$ is a $P^1$-bundle, and then $(K \otimes L^2)|_F = 0$. Therefore $R$ is a sum of fibres, since it belongs to $|K_X \otimes L^2|$. 

Added in proof. Question (2.3) has recently been given a positive answer by T. Fujita and P. Ionescu, independently.

REFERENCES