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Integral Equivalence of Two Systems of Differential Equations

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Equazioni differenziali ordinarie. — *Integral Equivalence of Two Systems of Differential Equations.* Nota di JAROSŁAW MORCHAŁO, presentata (*) dal Corrisp. R. CONTI.

RIASSUNTO. — Si studia l'equivalenza asintotica fra le soluzioni di un sistema lineare e quelle di una perturbazione non lineare. Vengono date condizioni sufficienti per l'esistenza di un omeomorfismo fra le soluzioni limitate di tali sistemi.

The object of this paper is the study of asymptotic equivalence between the solutions of the differential equations

$$y'(t) = A(t)\, y(t)\,, \tag{I}$$

and

$$x'(t) = A(t)\, x(t) + F(t\,, x(t)\,, (Tx)(t))\,. \tag{II}$$

By means of the contraction mapping principle, we prove the existence of a homeomorphism H between the sets of bounded solutions of (I) and (II). Moreover, we are going to investigate the $(g\,, p)$ integral equivalence between equations (I) and (II), such that to each bounded solution $x(t) = Hy(t)$ of (II) we have

$$|\, g^{-1}(t)\,[y(t) - Hy(t)]\,| \in L_p(t_0\,, \infty)\,.$$

Our results extend some theorems obtained by Haščak and Švec [2], which are proved using Schauder's fixed point theorem.

Let Y be a finite dimensional linear space with norm $|\,.\,|$; $\mathscr{A}$ – the algebra of linear functions from Y to Y with induced norm $\|\,.\,\|$; I – the identity in $\mathscr{A}$; A – a continuous function from $J = < t_0\,, \infty)$, $t_0 \geq 0$ into $\mathscr{A}$; U – the fundamental solution for (I), i.e. U is a continuously differentiable function from J to $\mathscr{A}$ such that $U(t_0) = I$, $U'(t) = A(t)\, U(t)$, $t \in J$, M_1 – a subspace of Y such that x belongs if the function from J to Y described by $t \to U(t)\, x$ is bounded, M_2 – a subspace of Y such that $Y = M_1 \oplus M_2$ and P_i – a supplementary projection in $\mathscr{A}$ i.e. $P_i Y = M_i$ $(i = 1\,, 2)$.

Let L be a function from $J \times J$ to $\mathscr{A}$ such that

$$L(t\,, s) = \begin{cases} U(t)\, P_1\, U^{-1}(t) & \text{if} \quad s \leq t\,, \\ -U(t)\, P_2\, U^{-1}(s) & \text{if} \quad s > t\,. \end{cases}$$

(*) Nella seduta del 26 gennaio 1985.

Let $F : J \times Y \times Y \to Y$ satisfy the Caratheodory conditions i.e. $F(t, u, v)$ is measurable in $t \in J$ for all $(u, v) \in Y \times Y$ and continuous in u, v for all $t \in J$, $T : C(J) \to Y$, where $C(J) = \{u : u \in C[J, Y]\}$. A function $q : \langle 0, \infty) \to Y$, is said to be in $L_p(0, \infty)$, $(p > 0)$ if $\int_0^\infty | q(t) |^p dt < \infty$, where $|.|$ is any Y-norm.

Let $g(t)$ be a continuous $n \times n$ matrix such that $g^{-1}(t)$ exists for all $t \in J$.

DEFINITION 1. *We will say that a function z is g-bounded on the interval* J *if* $\sup \{| g^{-1}(t) z(t) | < \infty, t \in J\}$:

DEFINITION 2. *We shall say that two systems* (I) *and* (II) *are* (g, p) $(p \geq 1)$ *integral equivalent on* J *if for each solution y of* (I) *there exists a solution x of* (II) *such that*

$$\text{(III)} \qquad | g^{-1}(t) [x(t) - y(t)] | \in L_p(t_0, \infty),$$

and conversely for each solution x of (II) *there exists a solution y of* (I) *such that* (III) *holds.*

Let C be the set of all continuous functions with domain J and range Y. A Cg- space is the set of all continuous functions x in Y, such that

$$\| x \|_g = \sup \{| g^{-1}(t) x(t) | \leq M < \infty, 0 < M = \text{const.}\ t \in J\}.$$

The following Lemmas will be used in our subsequent discussion:

LEMMA 1 [1]. *Let* C *be the Banach space of bounded continuous functions* x: $J \to Y$ *with the norm* $\| x \| = \sup \{| x(t) |, t \in J\}$.

Let $G : C \to C$ *be a contraction and* V_1, V_2 *non-empty subsets of* C *such that* $(I - G) V_2 \subset V_1$. *If* $H : V_1 \to V_2$ *satisfies the relation* $Hy(t) = y(t) + GHy(t)$, $t \in J$, $y \in V_1$, *then* H *is a homeomorphism of* V_1 *into* V_2.

LEMMA 2 [2]. *Let* $f(t) \geq 0$ *be continuous on* $\langle 0, \infty)$ *and such that*

$$\int_0^\infty s f(s)\, ds < \infty, \qquad \textit{then} \qquad \int_t^\infty f(s)\, ds \in L_p\ (0, \infty), (p \geq 1).$$

LEMMA 3. *If*

1) *$g(t)$ is a continuous $n \times n$ matrix such that $g^{-1}(t)$ exists for all* $t \in \langle 0, \infty)$,

2) *$\varphi(t)$ it a positive function for* $t \in \langle 0, \infty)$,

3) *$U(t)$ is a non-singular matrix for all* $t \in \langle 0, \infty)$,

4) P *is a projection in* $\mathscr{A}$,

$$5)\ \left(\int_0^t \| g^{-1}(t)\, U(t)\, PU^{-1}(s)\, \varphi(s) \|^q\, ds\right)^{1/q} \leq K < \infty\,,\, t \in (0, \infty),\, (q \geq 1)\,,$$

$$\int_0^\infty \exp\left(-K^{-q} \int_0^t (\varphi(s))^q\, \| g(s) \|^{-q}\, ds\right) dt < \infty\,,\, (p + q = pq)\,,$$

then $\lim\limits_{t \to \infty} \| g^{-1}(t)\, U(t)\, P \| = 0$ *and* $\| g^{-1}(t)\, U(t)\, P \| \in L_p(0, \infty)$, $(p \geq 2)$.

The proof of the Lemma is omitted since it follows Haščak's and Švec's proof [[2], Lemma 3] in a straightforward manner (because of $h(t) = (\varphi(t))^q \|\, \| U(t)\, P \|^{-q}$).

THEOREM 1 [3]. *Suppose that* Z *is a mapping of a complete metric space* $\langle X, d \rangle$ *into itself and* $d(Z(x), Z(y)) \leq q_0(a, b)\, d(x, y)$ *for each* $(x, y) \in X$ *such that* $a \leq d(x, y) \leq b$, $q_0(a, b) < 1$, $(0 < a \leq b)$.

Then there exists a unique $u \in X$ *such that* $u = Z(u)$.

MAIN RESULTS

THEOREM A. *If*:

1) $r : J_0 \times J_0 \to J\ (J_0 = \langle 0, \infty))$ *is a non-decreasing function with respect to each variable separately and such that*

$$\sup \left\{ \frac{r(u, v)}{\max(u, v)}\,,\, a \leq u\,,\, v \leq b\,,\, 0 < a \leq b \right\} < 1\,,$$

2) *there exist supplementary projections* $P_i\ (i = 1, 2)$ *in* $\mathscr{A}$ *and constant* $K > 0$ *such that*

$$\left(\int_0^t \| g^{-1}(t)\, U(t)\, P_1\, U^{-1}(s) \|^q\, ds\right)^{1/q} + \left(\int_t^\infty \| g^{-1}(t)\, U(t)\, P_2\, U^{-1}(s) \|^q\, ds\right)^{1/q} \leq$$

$\leq K < \infty$ *for all* $t \in J_0$,

3) *there exist continuous functions* $h \in L_p(J_0, R^+)$ *and positive constants* q, α, K_1 *such that*

$$| F(t, u, v) - F(t, u_1, v_1) | \leq h(t)\, r(| g^{-1}(t)\, [u(t) - u_1(t)] |\,,\, | v(t) - v_1(t) |)$$

where

$$| v(t) - v_1(t) | \leq \alpha \, | g^{-1}(t) [u(t) - u_1(t)] | , \; 0 < \alpha \leq 1 , \; u, v, u_1, v_1 \in Y,$$

$$\int_0^\infty h^p(t) \, dt < \infty , \; K \Big(\int_0^\infty h^p(t) \, dt \Big)^{1/p} = K_1 \leq 1 , \; p + q = pq , \; (p \geq 2) ,$$

$$4) \quad \int_0^\infty | F(t, 0, 0) |^p \, dt < \infty ,$$

then there exists a homeomorphism H *from the set of g-bounded solutions of* (I) *into the g-bounded solutions of* (II).

Proof. Let y be a g-bounded solution of (I) on J. Then there exists a constant $a > 0$ such that $y \in B_{g,a}$, where

$$B_{g,a} = \Big\{ z : z \in C[J, Y] \Big\} \text{ and } \sup \Big\{ | g^{-1}(t) z(t) | \leq a , t \in J \Big\} .$$

Define, for $x \in B_{g,2a}$, the operator R

$$\text{(1)} \qquad Rx(t) = y(t) + \int_{t_0}^{t} U(t) P_1 U^{-1}(s) F(s, x(s), (Tx)(s)) \, ds -$$

$$- \int_t^\infty U(t) P_2 U^{-1}(s) F(s, x(s), (Tx)(s)) \, dt , \; t \in J .$$

Then

$$| g^{-1}(t) Rx(t) | \leq a + \int_{t_0}^{t} \| g^{-1}(t) U(t) P_1 U^{-1}(s) \| h(s) r(| g^{-1}(s) x(s) | ,$$

$$| g^{-1}(s) x(s) |) \, ds + \int_t^\infty \| g^{-1}(t) U(t) P_2 U^{-1}(s) \| h(s) r(| g^{-1}(s) x(s) | ,$$

$$| g^{-1}(s) x(s) |) \, ds + \int_{t_0}^{t} \| g^{-1}(t) U(t) P_1 U^{-1}(s) \| \, | F(s, 0, 0) | \, ds +$$

$$+ \int_t^\infty \| g^{-1}(t) U(t) P_2 U^{-1}(s) \| \, | F(s, 0, 0) | \, ds \leq$$

$$\leq a + r(2a, 2a)\left\{\left(\int_{t_0}^{t} \| g^{-1}(t) \, U(t) \, P_1 \, U^{-1}(s) \|^q \, ds\right)^{1/q} \left(\int_{t_0}^{t} h^p(s) \, ds\right)^{1/p} + \right.$$

$$\left. + \left(\int_{t}^{\infty} \| g^{-1}(t) \, U(t) \, P_2 \, U^{-1}(s) \|^q \, ds\right)^{1/q} \left(\int_{t}^{\infty} h^p(s) \, ds\right)^{1/p}\right\} +$$

$$+ \left(\int_{t_0}^{t} \| g^{-1}(t) \, U(t) \, P_1 \, U^{-1}(s) \|^q \, ds\right)^{1/q} \left(\int_{t_0}^{t} | F(s, 0, 0) |^p ds\right)^{1/p} +$$

$$+ \left(\int_{t}^{\infty} \| g^{-1}(t) \, U(t) \, P_2 \, U^{-1}(s) \|^q \, ds\right)^{1/q} \left(\int_{t}^{\infty} | F(s, 0, 0) |^p ds\right)^{1/p}.$$

If we choose t_0 such that

$$r(2a, 2a) \, K \left(\int_{t_0}^{t} h^p(s) \, ds\right)^{1/p} \leq \frac{a}{2} \quad \text{and} \quad K \left(\int_{t_0}^{t} | F(s, 0, 0) |^p ds\right)^{1/p} \leq \frac{a}{2}, \; t \geq t_0,$$

we have that R maps $B_{g,2a}$ into itself. Now we are going to demonstrate using the Krasnosielski's Theorem 1, that the operator R has a unique fixed point in $B_{g,2a}$. For $x_1, x_2 \in B_{g,2a}$, we have

$$| g^{-1}(t) \, [Rx_1(t) - Rx_2(t)] | \leq \int_{t_0}^{t} \| g^{-1}(t) \, U(t) \, P_1 \, U^{-1}(s) \| \, .$$

$$| F(s, x_1(s), (Tx_1)(s)) - F(s, x_2(s), (Tx_2)(s)) | \, ds +$$

$$+ \int_{t}^{\infty} \| g^{-1}(t) \, U(t) \, P_2 \, U^{-1}(s) \| \, | F(s, x_1(s), (Tx_1)(s)) -$$

$$- F(s, x_2(s), (Tx_2)(s)) | \, ds \leq$$

$$\leq K \left(\int_{t_0}^{\infty} h^p(s) \, ds\right)^{1/p} r(\| x_1 - x_2 \|_g, \| x_1 - x_2 \|_g) \text{ for all } x_1, x_2 \in B_{g,2a}.$$

Hence $\| Rx_1 - Rx_2 \|_g \leq r(\| x_1 - x_2 \|_g, \| x_1 - x_2 \|_g)$. Thus we can apply Theorem 1 which yields the existence of a unique $x \in B_{g,2a}$ such that $x = Rx$.

An easy computation shows that the fixed point $x(t) = Rx(t)$, $t \in J$ is a solution of (II).

Let $B_{g,I}$ and $B_{g,II}$ denote the spaces of g-bounded solutions of (I) and (II) respectively. We define the mapping $H : B_{g,I} \to B_{g,II}$ in the following way: for every $y \in B_{g,I}$, Hy will be the fixed point of the contraction R. Thus for $t \in J$, $Hy = RHy(t)$.

According to Lemma 1, where $V_1 = B_{g,I}$ and $V_2 = B_{g,II}$, H is a homeomorphism from $B_{g,I}$ to $B_{g,II}$ and the inverse mapping is

$$H^{-1} x(t) = x(t) - R_1 x(t) \ , \ x \in B_{g,II} \ , \ t \in J \ ,$$

where

$$R_1 x(t) = \int_{t_0}^{t} U(t) P_1 U^{-1}(s) F(s, x(s), (Tx)(s)) \, ds -$$

$$- \int_{t}^{\infty} U(t) P_2 U^{-1}(s) F(s, x(s), (Tx)(s)) \, ds .$$

THEOREM B. *If:*

1) *the assumptions of Theorem* A *hold,*

2) $\int_0^\infty \| P_1 U^{-1}(s) \| h(s) \, ds < \infty \quad , \quad \int_0^\infty \| P_1 U^{-1}(s) \| \, | F(s, 0, 0) | \, ds < \infty ,$

3) $\int_0^\infty s h^p(s) \, ds < \infty \quad , \quad \int_0^\infty s \, | F(s, 0, 0) |^p ds < \infty ,$

4) $\int_0^\infty \exp\left(- K^q \int_0^t \| g(s) \|^{-q} ds\right) dt < \infty ,$

then $| g^{-1}(t) [Hy(t) - y(t)] | \in L_p(t_0, \infty)$.

Proof. From (1) and hypothesis of the Theorem B we have

$$| g^{-1}(t) [Hy(t) - y(t)] | \leq \int_{t_0}^{t} \| g^{-1}(t) U(t) P_1 U^{-1}(s) \| \, | F(s, Hy(s),$$

$$(THy)(s)) | \, ds + \int_{t}^{\infty} \| g^{-1}(t) U(t) P_2 U^{-1}(s) \| \, | F(s, Hy(s), (THy)(s)) | \, ds \leq$$

(2)
$$\leq \| g^{-1}(t) U(t) P_1 \| \left\{ r(2a, 2a) \int_{t_0}^{t} \| P_1 U^{-1}(s) \| h(s) \, ds + \int_{t_0}^{t} \| P_1 U^{-1}(s) \| \right.$$

$$\left. | F(s, 0, 0) | \, ds \right\} + r(2a, 2a) \int_{t}^{\infty} \| g^{-1}(t) P_1 U^{-1}(s) \| h(s) \, ds + \int_{t}^{\infty} \| g^{-1}(t) U(t)$$

$$P_2 U^{-1}(s) \| \, | F(s, 0, 0) | \, ds .$$

Hence

$$\| g^{-1}(t)\, U(t)\, P_1 \| \{ r(2a, 2a) \int_{t_0}^{t} \| P_1 U^{-1}(s) \| h(s)\, ds + \int_{t_0}^{t} \| P_1 U^{-1}(s) \|$$

$$| F(s, 0, 0) |\, ds \} \leq \| g^{-1}(t)\, U(t)\, P_1 \| \{ r(2a, 2a) \int_{t_0}^{\infty} \| P_1 U^{-1}(s) \| h(s)\, ds +$$

$$+ \int_{t_0}^{\infty} \| P_1 U^{-1}(s) \| \, | F(s, 0, 0) |\, ds \} .$$

Since (from Lemma 3) $\| g^{-1}(t)\, U(t)\, P_1 \| \in L_p(t_0, \infty)$, it is evident that this first term in the above inequality belongs to $L_p(t_0, \infty)$.

Taking into account the second term of this inequality, we obtain

$$r(2a, 2a) \int_{t}^{\infty} \| g^{-1}(t)\, U(t)\, P_1 U^{-1}(s) \| h(s)\, ds + \int_{t}^{\infty} \| g^{-1}(t)\, U(t)\, P_2 U^{-1}(s) \|$$

$$F(s, 0, 0) |\, ds \leq r(2a, 2a) \left(\int_{t}^{\infty} \| g^{-1}(t)\, U(t)\, P_1 U^{-1}(s) \|^{q} ds \right)^{1/q} \left(\int_{t}^{\infty} h^{p}(s)\, ds \right)^{1/p} +$$

$$+ \left(\int_{t}^{\infty} \| g^{-1}(t)\, U(t)\, P_2 U^{-1}(s) \|^{q} ds \right)^{1/q} \left(\int_{t}^{\infty} | F(s, 0, 0) |^{p} ds \right)^{1/p} \leq$$

$$\leq K r(2a, 2a) \left(\int_{t}^{\infty} h^{p}(s)\, ds \right)^{1/p} + K \left(\int_{t}^{\infty} | F(s, 0, 0) |^{p} ds \right)^{1/p} .$$

Also, from 3) and Lemma 2, this second term belongs to $L_p(t_0, \infty)$. The proof of the theorem is complete.

THEOREM C. *If*

1) *the assumptions of Theorem* A *hold,*

2) $\lim_{d \to \infty} \left(\int_{d}^{\infty} | b(t) |^{p} dt \right)^{1/p} = 0$ *for all* $b \in L_p(t_0, \infty)$,

3) $\int_{0}^{\infty} \exp\left(-K^{q} \int_{0}^{t} \| g(s) \|^{-q} ds\right) dt < \infty$,

then $\lim_{t \to \infty} | g^{-1}(t)\, [Hy(t) - y(t)] | = 0$.

Proof. Let

$$(S\,b)\,(t) = \int_{t_0}^{t} U\,(t)\,P_1\,U^{-1}\,(s)\,b\,(s)\,ds - \int_{t}^{\infty} U\,(t)\,P_2\,U^{-1}\,(s)\,b\,(s)\,ds\,,$$

$$(S_g\,b)\,(t) = g^{-1}\,(t)\,(Sb)\,(t) = \int_{t_0}^{t} g^{-1}\,(t)\,U\,(t)\,P_1\,U^{-1}\,(s)\,b\,(s)\,ds -$$

$$- \int_{t}^{\infty} g^{-1}\,(t)\,U\,(t)\,P_2\,U^{-1}\,(s)\,b\,(s)\,ds\,.$$

The assumption 1) of Theorem implies that

$$\|\,(Sb)\,\|_g \leq K\,\|\,b\,\|_p \quad \text{for all} \quad b \in L_p\,(t_0\,,\infty)\,.$$

For any $\tau > t_0$ let us put

$$u_\tau = S_g\,(\chi_{<t_0,\tau>}\,F\,(\cdot\,,Hy\,,(THy)))\;\;,\;\; v_\tau = S_g\,(\chi_{<\tau,\infty>}\,F\,(\cdot\,,Hy\,,(THy)))\,.$$

Because

$$|\,\chi_{<\tau,\infty)}\,(t)\,F\,(t\,,Hy\,(t)\,,(THy)\,(t))\,| \leq |\,\chi_{<\tau,\infty)}\,(t)\,F\,(t\,,0\,,0)\,| +$$
$$+ \chi_{<\tau,\infty)}\,(t)\,h\,(t)\,r\,(\|\,Hy\,\|_g\,,\|\,Hy\,\|_g) \;\text{ for }\; t \in J\,,$$

then

$$\|\,v_\tau\,\|_g \leq Kr\,(\|\,Hy\,\|_g\,,\|\,Hy\,\|_g)\left(\int_\tau^\infty h^p\,(t)\,dt\right)^{1/p} + K\left(\int_\tau^\infty |\,F\,(t\,,0\,,0)\,|^p dt\right)^{1/p}.$$

By assumption 2) $\lim\limits_{\tau\to\infty}\left(\int_\tau^\infty h^p\,(t)\,dt\right)^{1/p} = 0$ and $\lim\limits_{\tau\to\infty}\left(\int_\tau^\infty |\,F\,(t\,,0\,,0)\,|^p\,dt\right)^{1/p} = 0$.

Therefore for any $\varepsilon > 0$ we can choose $\tau > t_0$ such that $\|\,v_\tau\,\|_g \leq \frac{\varepsilon}{2}$.

Moreover, by 1), 3) and Lemma 3, $\lim\limits_{t\to\infty}\|\,g^{-1}\,(t)\,U\,(t)\,P_1\,\| = 0$.

Hence there exists a $t_1 > t_0$ such that

$$|\,u_\tau\,(t)\,| \leq \|\,g^{-1}\,(t)\,U\,(t)\,P_1\,\| \int_{t_0}^{\tau} |\,P_1\,U^{-1}\,(s)\,F\,(s\,,Hy\,(s)\,,(THy)\,(s))\,|\,ds \leq \frac{\varepsilon}{2}\,,$$

for all $t \geq t_1$. From this by (1) it follows that

$$g^{-1}(t)\,[\mathrm{H}y(t) - y(t)] = \int_{t_0}^{t} g^{-1}(t)\,\mathrm{U}(t)\,\mathrm{P}_1\mathrm{U}^{-1}(s)\,\mathrm{F}(s\,,\mathrm{H}y(s)\,,(\mathrm{TH}y)(s))\,\mathrm{d}s\, -$$

$$-\int_{t}^{\infty} g^{-1}(t)\,\mathrm{U}(t)\,\mathrm{P}_2\,\mathrm{U}^{-1}(s)\,\mathrm{F}(s\,,\mathrm{H}y(s)\,,(\mathrm{TH}y)(s))\,\mathrm{d}s = u_\tau(t) + v_\tau(t)\,.$$

Hence $|g^{-1}(t)\,[\mathrm{H}y(t) - y(t)]| \leq |u_\tau(t)| + |v_\tau(t)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$

As $\varepsilon > 0$ is arbitrary this implies $\lim_{t\to\infty} |g^{-1}(t)\,[\mathrm{H}y(t) - y(t)]| = 0.$ This completes the proof of the Theorem.

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