Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A remark on complex powers of analytic functions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **78** (1985), n.1-2, p. 1–3.

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<http://www.bdim.eu/item?id=RLINA_1985_8_78_1-2_1_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1985.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Sedute del 26 gennaio e del 9 febbraio 1985

Presiede il Presidente della Classe GIUSEPPE MONTALENTI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *A remark on complex powers of analytic functions.* Nota (*) di GIUSEPPE ZAMPIERI, presentata dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Sia $K \Subset \mathbf{R}^n$ un compatto, $f \geq 0$ una funzione analitica all'intorno di K, ed m la massima molteplicità in K degli zeri di f; si prova che la potenza f^λ ($\lambda \in \mathbf{C}$, $\operatorname{Re} \lambda > -1/m$) è integrabile in K. L'estensione meromorfa dell'applicazione $\lambda \to f^\lambda$ da $\operatorname{Re} \lambda > 0$ a tutto $\mathbf{C}$ (con valori in $\mathscr{D}'(K)$ anziché in $L^1(K)$) era già stata provata in [1] e [2].

1. In [1] and [2] it was proven that the power f^λ (f being a non-negative analytic function in R^n and λ a complex number), which belongs to L^1_{loc} for $\operatorname{Re} \lambda > 0$, extends analytically to be a distribution meromorphically depending on λ with rational negative poles.

Here we want to give a very elementary proof of the extension up to $\operatorname{Re} \lambda = 0$ with L^1_{loc} values:

PROPOSITION. *For* $K \Subset \mathbf{R}^n$ *let* m *denote the largest multiplicity of the roots of* f *which occur in* K. *Then* $f^\lambda \in L^1$ *on a neighbourhood of* K *for* $\operatorname{Re} \lambda > -1/m$.

(*) Pervenuta all'Accademia il 25 settembre 1984.

REMARK. Let $\mathcal{D}'(K)$, $(L^1(K))$, be the spaces of distributions, (integrable functions), on a neighbourhood of K. Let α be the first pole of the mapping from $\mathbf{C}$ to $\mathcal{D}'(K) : \lambda \to f^\lambda$, and set $\beta = \inf\{\operatorname{Re}\lambda : f^\lambda \in L^1(K)\}$. We conjecture here, under suggestion of Atiyah's proof, that $\alpha = \beta$. In such case we would then obtain from the Proposition and from the trivial fact that $f^\lambda \notin L^1(K)$ for $\operatorname{Re}\lambda < -n/m$:

$$-n/m \leq \alpha \leq -1/m .$$

Note that $\alpha = -n/m$ for $f(\xi) = \sum_{i=1}^{n} \xi_i^m$ (m even) and $\alpha = -1/m$ for $f(\xi) = \xi_1^m$.

2. PROOF OF THE PROPOSITION

The proposition having local character, we will prove it in a complex neighbourhood U of 0. We will assume that f is analytic in U with $f(\zeta) = 0(|\zeta|^m)$ for $\zeta \to 0$ and $\sum_{|\alpha|=m} |f^{(\alpha)}(0)| \neq 0$. Thus for suitable $\eta \in \mathbf{R}^n$, $|\eta| = 1$, we have $|\sum_{|\alpha|=m} f^{(\alpha)}(0)\,\eta^\alpha| \neq 0$.

By change of the coordinate system let $\eta = (0, \dots, 1)$. If then we denote by ζ' the first $n-1$ variables and by ζ_n the last one and choose a suitably small U, we have the factorization

$$f(\zeta) = h(\zeta) \prod_{j=1}^{m} (\zeta_n - \mu_j(\zeta')) , \qquad \zeta \in U ,$$

where $h(0) = \dfrac{\partial^m f}{\partial \zeta_n^m}(0) / m! \neq 0$ and $|\mu_j(\zeta')| = 0(|\zeta'|)$.

(*a*) First we have

$$c\,|f(\zeta)|^{1/m} \geq \inf_j |\zeta_n - \mu_j(\zeta')| , \qquad \zeta \in U \tag{2.1}$$

provided hat U is so small that $|h(\zeta)| \geq c^{-m}\ \forall \zeta \in U$.

(*b*) Let us put now $g(\zeta', \zeta_n) = \prod_{j=1}^{m} (\zeta_n - \mu_j(\zeta'))$ and write $g(\zeta', \zeta_n) = \zeta_n^m + a_1(\zeta')\,\zeta_n^{m-1} + a_2(\zeta')\,\zeta_n^{m-2} + \dots$ with $|a_j(\zeta')| = 0(|\zeta'|^{m-j})$. Let U' be the projection of U on the ζ'-plane; we then have $|a_j(\zeta') - a_j(\eta')| \leq c\,|\zeta' - \eta'|$, $\zeta', \eta' \in U'$ (for the coefficients a_j can be supposed to be regular in a larger set than U'). Thus we can write $g(\zeta', \zeta_n) = g(\eta', \zeta_n) + r(\zeta', \eta', \zeta_n)$ where r has degree $\leq m-1$ in ζ_n and coefficients uniformly bounded by $c\,|\zeta' - \eta'|$. We then have under suitable labelling ([4])

$$(2.3)\qquad |\mu_j(\zeta') - \mu_j(\eta')| \leq c' \, |\zeta' - \eta'|^{1/m}, \qquad j = 1, \dots, m.$$

(*c*) Let us cover $U'_{\mathbf{R}} = U' \cap \mathbf{R}^{n-1}$ by a family of spheres B_i, $i = 1, \dots, M$ with centres ξ^i and radii ε in such a way that $M = M_\varepsilon \leq c\, \varepsilon^{-n+1}$. Let us consider in U the points $(\xi^i, \mu_j(\xi^i))$, $j = 1, \dots, m$, over each ξ^i, and the family of polycylinders in $\mathbf{R}^{n-1} \times \mathbf{C} : B_{i,j} = \{|\xi' - \xi^i| < \varepsilon\} \times \{|\zeta_n - \mu_j(\xi^i)| < c\, \varepsilon^{1/m}\}$. We claim that for suitable choice of c

$$(2.4)\qquad \bigcup_{i,j} B_{i,j} \supseteq \{\xi \in U_{\mathbf{R}} : f(\xi) < \varepsilon\}.$$

In fact if $f(\xi) < \varepsilon$, then $|\xi_n - \mu_j(\xi')| < c\, \varepsilon^{1/m}$ for some j due to (2.1). Since $|\xi' - \xi^i| < \varepsilon$ for some i, then $|\mu_j(\xi') - \mu_j(\xi^i)| < c'\, \varepsilon^{1/m}$ due to (2.3), and therefore $|\xi_n - \mu_j(\xi^i)| < c\, \varepsilon^{1/m} + c'\, \varepsilon^{1/m} = c''\, \varepsilon^{1/m}$. This proves (2.4).

(*d*) We have obviously

$$(2.5)\qquad \mathrm{Vol}\left(\bigcup_{i,j} (\{|\xi' - \xi^i| < \varepsilon\} \times \{|\xi_n - \mu_j(\xi^i)| < c\, \varepsilon^{1/m}\})\right) \leq c' M_\varepsilon\, \varepsilon^{n-1+1/m} = c''\, \varepsilon^{1/m} \ (\mathrm{Vol} = \text{volume}).$$

(*e*) Taking into account (2.4) and (2.5) we have for $b < 1/m$, $\varepsilon = 2^{-j}$ $(j \geq N >> 0)$

$$\int_{U_{\mathbf{R}}} f(\xi)^{-b}\, d\xi \leq \int_{U_{\mathbf{R}} \cap \{f(\xi) > 2^{-N}\}} 2^{Nb}\, d\xi + \sum_{j=N}^{\infty} \int_{U_{\mathbf{R}} \cap \{2^{-(j+1)} \leq f(\xi) \leq 2^{-j}\}} 2^{(j+1)b}\, d\xi$$

$$\leq c\, 2^{Nb} + \sum_{j=N}^{\infty} c'\, 2^{(j+1)b - j/m} < \infty \ (\text{due to } b < 1/m).$$

The proof is complete.

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