Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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The superfocal subgroup

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **76** (1984), n.6, p. 330–333.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1984_8_76_6_330_0>

Teoria dei gruppi. — *The superfocal subgroup.* Nota di MARIAN DEAČONESCU, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Nel presente lavoro vengono dimostrati teoremi d'esistenza di p-complementi normali nei gruppi finiti.

1. INTRODUCTION

Gaschütz [2] introduced two Frattini like subgroups of a finite group G : L(G) = the intersection of the maximal self-normalizing subgroups of G and R (G) = the intersection of the maximal and normal subgroups of G.

While the structure of L (G) is well-known (see [1]), we know very little about the structure of R (G). We shall establish in this note some connections between the structure of R (G) and that of the whole group G by means of the so called superfocal subgroup of an S_p-subgroup of G. Results like Grün's theorems hold for the superfocal subgroup, leading to sufficient conditions for the existence of normal p-complements.

2. PRELIMINARIES

The only groups considered here are finite. Any notation not explicitly defined conforms to that of [3]. In what follows, G will be a finite group and P will be a fixed S_p-subgroup of G $(P \neq 1)$. The letter M will be reserved for the maximal subgroups of G. The intersection of the subgroups H of G having the property $\mathscr{P}$ will be denoted by $\cap\{H \mid \mathscr{P}\}$. If H is a subgroup of G we shall denote by H_G^0 the intersection of the maximal subgroups of G which contain H. With these conventions, we define $R(G) = (G')_G^0 = \cap\{M \mid M \lhd G\}$. It is clear that R (G) is a characteristic subgroup of G and that $R(G)/G' = \Phi(G/G')$.

We shall also consider $R_{p'}(G)$ and $R_p(G)$ which are the intersections of the maximal and normal subgroups of G of index prime with p and of index p respectively. Then $R(G) = R_{p'}(G) \cap R_p(G)$, with $R_p(G)/R(G)$ a p'-group and $R_{p'}(G)/R(G)$ a p-group.

The following elementary property of R (G) will be used without other specification: for every $H \leq G$, we have $R(H) \leq R(G)$.

(*) Nella seduta del 15 giugno 1984.

Let $H \leq K \leq G$; we say that H is *strongly closed* in K (with respect to G) if $K \cap H^x \leq H$ for every $x \in G$. We close this section with some well-known results:

2.1 (The first theorem of Grün). $P \cap G' = \langle P \cap N_G(P)', P \cap (P^x)', x \in G \rangle$.

2.2. THEOREM (Roquette, [5]). *If* $H \lhd G$ *and* $H \cap P \leq \Phi(P)$, *then* H *has a normal p-complement.*

2.3. THEOREM (Satz 4.8, p. 432 of [4]). G *has a normal p-complement if and only if* $P \cap G' \leq \Phi(P)$.

2.4. THEOREM (Gaschütz, [2]). *If* $H \lhd G$, *then* $\Phi(H) \leq \Phi(G)$.

3. THE SUPERFOCAL SUBGROUP

By analogy with the focal subgroup of P (see [3] for its main properties), we introduce the *superfocal subgroup* of P (with respect to G) to be $P \cap R(G)$. Since $G' \leq R(G)$, the superfocal subgroup contains the focal subgroup $P \cap G'$. The next result is trivial but important:

3.1. LEMMA. *i)* $R(N_G(P)) \leq N_G(P) \cap R(G)$.
ii) $\Phi(P) \leq P \cap R(G)$.

Having the notational convention of section 2 in mind, we can characterize the superfocal subgroup of P as follows:

3.2. THEOREM. $P \cap R(G) = (P \cap G')^0_P$.

Proof. Note that a subgroup Q of P with $|P : Q| = p$ is contained in a normal subgroup M of G with $|G : M| = p$ if and only if $P \cap G' \leq Q$. Indeed, if $Q \leq M$ and $|G : M| = p$, it follows that $G' \leq M$ and $P \cap G' \leq \leq P \cap M = Q$. Conversely, if $P \cap G' \leq Q$, we have that $QG' \cap P = = Q(P \cap G') = Q$ and so $p \mid |G : QG'|$. Since G/QG' is abelian, there exists a normal subgroup M of G such that $|G : M| = p$ and $QG' \leq M$, $Q = = P \cap M$.

Let now $R_1(G)$ be the intersection of the normal (maximal subgroups of G of index p. Then $P \cap R(G) = P \cap R_1(G) = P \cap (\cap \{M \mid M \lhd G, |G : M| = p\}) = \cap \{P \cap M \mid M \lhd G, |G : M| = p\} = \cap \{Q \mid |P : Q| = p, P \cap G' \leq Q < P\}$, i.e. $P \cap R(G) = (P \cap G')^0_P$.

A result like Theorem 7.3.1. of [3] can be stated for the superfocal subgroup, showing that G possesses a unique maximal elementary abelian p-factor group which is isomorphic to $P/P \cap R(G)$.

One can give another characterization of the superfocal subgroup, which is an analogue of the first theorem of Grün:

3.3. THEOREM. $P \cap R(G) = \langle P \cap R(N_G(P)), P \cap \Phi(P^x), x \in G \rangle$.

Proof. Let S be the right side of the statement, $R = R(G)$ and $N = N_G(P)$. It is clear that $S \lhd P$. We shall prove first that $S \leq P \cap R$.

By 3.1, $P \cap R(N) \leq P \cap R$. On the other hand, we have for every $x \in G$ that $P \cap \Phi(P^x) \leq P \cap R(N^x) \leq P \cap N^x \cap R$, which shows that $S \leq P \cap R$.

In order to prove that $P \cap R \leq S$, observe that, by 3.2, $P \cap R = (P \cap G')^0_P$. Since $P \cap N' \leq P \cap R(N)$ and since $P \cap (P^x)' \leq P \cap \Phi(P^x)$ for every $x \in G$, we obtain that $P \cap G' \leq S$. Thus $P \cap R = (P \cap G')^0_P \leq S^0_P$ and it remains to prove only that $S^0_P = S$. Since $\Phi(P) \leq S$, P/S is elementary abelian and consequently $\Phi(P/S) = 1$. This implies that $S^0_P/S = \Phi(P/S) = 1$, i.e., $S^0_P = S$. This ends the proof.

It is a simple exercise to prove, using 3.3 that $P \cap R(N_G(P)) = P \cap R(N_G(\Phi(P)))$. On the other hand, if $\Phi(P)$ is strongly closed in P, we obtain using 3.3 again that $P \cap R(G) = P \cap R(N_G(P))$. Summarizing, we state an analogue of the second theorem of Grün (see Th. 7.5.2. of [3]):

3.4. THEOREM. *If* $\Phi(P)$ *is strongly closed in* P, *then*

$$P \cap R(G) = P \cap R(N_G(P)) = P \cap R(N_G(\Phi(P))).$$

4. NORMAL p-COMPLEMENTS

The aim of this section is to use the above results in order to find sufficient conditions for a group to have a normal p-complement.

4.1. THEOREM. G *has a normal p-complement if and only if* $P \cap R(G) = \Phi(P)$.

Proof. If $P \cap R(G) = \Phi(P)$, then $P \cap G' \leq P \cap R(G) = \Phi(P)$ and G has a normal p-complement by 2.3. Conversely, if G has a normal p-complement, then $P \cap G' \leq \Phi(P)$ by 2.3 again and so $P \cap R(G) = (P \cap G')^0_P = \Phi(P)$.

In other words, G has a normal p-complement if and only if its maximal elementary abelian p-factor group is isomorphic to that of P.

4.2. COROLLARY. *If* $\Phi(P)$ *is strongly closed in P and if* $N_G(P)$ *has a normal p-complement, then* G *has a normal p-complement.*

Proof. Immediate, by 3.3 and 4.1.

4.3. COROLLARY. *Suppose that* $Z(P) \leq \Phi(P)$ *and that* $\Phi(P)$ *is strongly closed in* P. *If* $N_G(P)/C_G(P)$ *is a p-group, then* G *has a normal p-complement.*

Proof. Since $Z(P) \leq \Phi(P)$, we obtain $P \cap C_G(P) = Z(P) \leq \Phi(P)$ and 2.2 shows that $C_G(P)$ has a normal p-complement. Since $N_G(P)/C_G(P)$ is a p-group, it follows that $N_G(P)$ has a normal p-complement. The result is now a consequence of 4.2.

If R (G) has a normal p-complement, G, in general, has not one. The simplest example of such a group is that of S_3: it has not a normal 3-complement, but $R(S_3) = A_3$ has a (trivial) normal 3-complement.

Despite this situation, we can however establish.

4.4. THEOREM. *If* R (G) *and* $N_G(P)$ *both have normal p-complements, then* G *has a normal p-complement.*

Proof. Let H be the normal p-complement of R (G). It is characteristic in R (G) and therefore normal in $R_{p'}(G)$. Since $R_{p'}(G)/R(G)$ and $R(G)/H$ are p-groups, $R_{p'}(G)/H$ is also a p-group, which shows that H is the normal p-complement of $R_{p'}(G)$.

Since $P \leq R_{p'}(G)$, we have that $R_{p'}(G) = PH$ and the Frattini argument gives that $G = R_{p'}(G)\, N_G(P) = PHN_G(P) = HN_G(P) = HKP$, where K is the normal p-complement of $N_G(P)$. Since $H \lhd G$, HK is a p'-subgroup of G. On the other hand, $HK \lhd G$ since P normalizes H and K. This shows hat HK is the normal p-complement of G.

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