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A Note on Badiale’s Characterization of the $q$-Gamma Functions

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Matematica. — *A Note on Badiale's Characterization of the q-Gamma Functions*. Nota di Marino Badiale e Francis J. Sullivan, presentata (*) dal Socio G. Scorza Dragoni.

Riassunto. — Si precisano alcuni risultati del lavoro accennato nel titolo.

In [1] and [2] the first author proved a number of results characterizing the $q$-gamma functions by functional equations. Recall that

$$T_q(x) = \begin{cases} \frac{1 - x^q}{(1 - q)^{1-x}} \big( (1 - q^x) / q \big)_\infty / (q^x ; q)_\infty & \text{if } 0 \leq q < 1 \\ q(x) (q^{-1} ; q^{-1})_\infty (q - 1)^{1-x} / (q^{-x} ; q^{-1})_\infty & \text{if } q > 1. \end{cases}$$

In particular $T_0(x) = 1$, and since $T_q(x) \to \Gamma(x)$ as $q \to 1$ we set $\Gamma_1(x) := \Gamma(x)$.

In analogy with results of Chapter 6 of [0] Badiale established the following: (Corollary to Proposition 2, Page 50 of [2]).

Let $f(q, x)$ be a positive continuous real valued function defined for $0 \leq q \leq 1$, $0 < x$ such that $df/dx$ is continuous and such that

$$(1.1) \quad f(q, x + 1) = ((1 - q^x)(1 - q)) f(q, x)$$

and

\[(2.1) \quad f(q, nx) f(q^n, 1/n) \ldots f(q^n, (n - 1)/n) = f(q^n, x) f(q^n, x + 1/n) \ldots f\left(q^n, x + \frac{n - 1}{n}(1 + q + \ldots + q^{n-1})\right)\]

for some positive integer \(n\) and all \(q < 1\), \(n \geq 2\).

Then \(f(q, x) = k_q \Gamma_q(x)\) for some \(k_q\) depending only on \(q\).

The main purpose of this brief note is to point out that in fact \(k_q\) is an absolute constant independent of \(q\). This gives an even stronger analogy with the classical results of Artin for \(\Gamma(x)\).

The proof is similar to the last part of the proof of Proposition 1 of (1). Indeed, replacing \(f(q, x)\) with \(k_q \Gamma_q(x)\) and \(f(q^n, x)\) with \(k_{qn} \Gamma_{qn}(x)\) in (2.1) gives, after simplification:

\[k_q k_{qn}^{-1} = k_{qn}^n.\]

Now \(k_{qn} \neq 0\) by positivity of \(f(q, x)\), so cancellation gives \(k_q = k_{qn}\). Iteration then shows \(k_q = k_{qn} = \ldots = k_{n} = \ldots\). Hence by continuity of the quotient \(f(q, x)/\Gamma_q(x) = k_q\), as \(j \to \infty\), \(q^n \to 0\) and \(k_q = \ldots = k_{n} \to k_0\). Thus \(k_q = k_0\) for any \(q < 1\), and since \(k_q = k_{n+1} = \ldots\) we also have \(k_q = k_1\), again by continuity. This proves that \(k_q\) is independent of \(q\).

A very minor improvement is also possible in Proposition 1 of [1]. This result is notable since it imposes no requirement of continuity in \(q\) on the function \(f(q, x)\), but rather positivity of \(f(q, x)\) together with other conditions.

In this case if we restrict our attention to the domain \(0 < q < 1\) (or to the domain \(q > 1\)) we may weaken the condition (1.3) of [1]

\[f(q, x) > 0 \text{ for all } q \text{ and all } x\]

to

\[(1.3)^* \quad f(q, x) \neq 0 \text{ for all } q \text{ and all } x.\]

In fact, for \(0 < q < 1\) condition (1.3)* together with the existence of \(d^2 f/dx^2\), (1.2) and (1.4) of [1] imply (1.3). One sees this as follows. We use the terminology of [1]. Condition (1.4), that is, the fact that \(\lim_{q \to 1} f(q, x) = \Gamma(x)\) for some \(x_0\) and the known positivity of \(\Gamma(x)\) give a \(q_0 < 1\) such that \(f(q, x_0) > 0\) for \(1 > q \geq q_0\). Existence of \(d^2 f/dx^2\) shows that \(f(q, x)\) is of constant sign for \(q\) fixed and variable \(x\), so that \(f(q, x) > 0\) for \(1 > q \geq q_0\) and all \(x\). Then (1.2) of [1] (which is our (2.1) with \(n = 2\)) gives \(f(q^2, x) > 0\) whenever \(f(q, x) > 0\), so that \(f(q^2, x) > 0\) whenever \(f(q, x) > 0\). Hence \(f(q, x) > 0\) for all \(q, 1 > q > 0\) and all \(x > 0\), and similarly one may treat the domain \(q > 1\). For \(q = 1\), it follows from Artin’s characterization of \(\Gamma(x)\) that \(f(1, x) = \pm \Gamma(x)\).
REFERENCES