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Nodal curves in $\mathbb{P}_3(\mathbb{C})$

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Geometria algebrica. — *Nodal curves in* $\mathbf{P}_3$ (**C**). Nota di EDOARDO BALLICO (*) e PAOLO OLIVERIO (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Siano d , g , t interi con $0 \leq t \leq g$; se esiste in $\mathbf{P}_3$ (**C**) una curva connessa, non singolare di grado d e genere g, allora esiste in $\mathbf{P}_3$ (**C**) una curva irriducibile di grado d, genere aritmetico g e t nodi.

INTRODUCTION

We work over the field **C** of complex numbers. In this note we prove the following theorem:

THEOREM. *For every* (d , g) *integers such that there exists a nonsingular connected curve* C *in* $\mathbf{P}_3$ *of genus* g *and degree* d, *and any* $0 \leq t \leq g$, *there exists in* $\mathbf{P}_3$ *an irreducible curve of genus* g, *degree* d *with exactly* t *nodes as only singularities.*

The proof of the theorem uses Severi-Wahl-Tannenbaum's theory of nodes in rational surfaces ([8] or [1] chap. 9) and the resolution by Gruson-Peskine [5] of a very old conjecture by Halphen, which describes exactly the integers (d , g) such that $\mathbf{P}_3$ contains a connected non-singular curve of degree d and genus g: the only restriction is given by Castelnuovo's refined bounds [4].

Gruson-Peskine prove the existence of such curves (apart from the trivial cases of plane curves and curves contained in a quadric) in non-singular cubic surfaces and in quartic surfaces with a double line, images by a birational morphism of a plane blown-up at 9 general points. The case of plane curves was considered by Severi and in a modern language by Tannenbaum [7], [8], who considered also the case of curves on a geometrically ruled rational surface, though not stating his results in the form we need ([2], [7], [8]).

It seems worthwhile to point out that even the case of a quadric, the trivial case, has as a corollary a slight improvement of a result of M. Raynaud [6] 2.1 about rank-3 semi-stable vector bundles on a general hyperelliptic curve.

The case of the cubic surface is very easy and short.

For the quartic there are some problems, since the anticanonical line bundle is not ample and thus we cannot always apply directly Wahl's criterion [9], used by Tannenbaum. However, a local computation [1] 9.59 permits us to

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resolve the difficulty. Furthermore, we construct our curves with nodes in a slightly different quartic surface.

This note is not self-contained. Wahl's results are stated, used and explained in [8]. The best geometric introduction to this theory is [1], Chapter 9.

§ 1. Geometrically ruled surfaces

A straightforward calculation and [8], 2.2 and 2.13 gives the following

PROPOSITION 1. *Let* F *be a geometrically ruled rational surface with invariant* $-e \leq 0$. *Let* $L \in \mathrm{Pic}(F)$, $L = (a, b)$ *with* $a > 0$, $b > ae$; *let* D *be a connected, smooth curve in* L *with genus* g *and* $0 \leq t \leq g$. *Then there exists* $C \in |D|$ *irreducible with only* t *nodes as singularities.*

If we apply the proposition above to the easiest case $e = 0$ (a non-singular quadric in $\mathbf{P}_3$) we obtain the existence of nodal curves contained in a quadric, as stated in the theorem. Furthermore, we see that such a nodal curve is a degeneration of non-singular curves in a fixed linear system. For $a = 2$, $b = g + 1$ we see that there exists a rational curve with g nodes degeneration of a family of non-singular, connected, hyperelliptic curves of genus g.

The simple degenerations in a quadric and the proof of [6], 2.1, give a small improvement of [6], 2.1, if $ch(k) = 0$. First recall M. Raynaud's definition.

DEFINITION. Let C be a non-singular projective curve. A semi-stable vector bundle E on C satisfies the condition (*) if for a general $L \in \mathrm{Pic}^0(C)$, $h^0(C, E \otimes L) = \max(0, \chi(E))$.

COROLLARY. *Over the complex field, on a general hyperelliptic curve of genus* $g \geq 2$ *every semi-stable vector bundle of rank* 3 *satisfies the condition* (*).

§ 2. The cubic surface

Let S be the cubic surface in $\mathbf{P}_3$, obtained blowing-up 6 points $P_1, \ldots, P_6$ in general position in $\mathbf{P}_2$.

We have $\mathrm{Pic}(S) \simeq \mathbf{Z}^7$ and a basis is the pull-back of a line in $\mathbf{P}_2$ and the exceptional divisors; in this basis $L \in \mathrm{Pic}(S)$ corresponds to $(a, m_1, \ldots, m_6) \in \mathbf{Z}^7$. By [5] we can choose 6 disjoint exceptional divisors which represent S as the blow-up of $\mathbf{P}_2$ at 6 points in such a way that, for fixed L, we have $a \geq m_1 + m_2 + m_3$ and $m_1 \geq m_2 \geq \ldots \geq m_6$; we call such a basis adequate (for L). In an adequate basis L is generated by global sections if and only if $m_6 \geq 0$ and in this case a general section is smooth and it is connected except for $(n, n, 0, \ldots, 0)$ with $n \geq 2$.

PROPOSITION 2. *Let* $L \in \mathrm{Pic}(S)$ *be as above with* $m_6 \geq 0$ *and let* D *be the zero locus of a general section of* L, D *smooth, connected of genus* g *and degree* d; *take* $0 \leq t \leq g$. *Then in* | D | *there is an irreducible curve with t nodes as only singularities.*

Proof. By Wahl's criterion [8], 2.2 and 2.13, since $-\omega_s$ is ample, it is sufficient to show that in | D | there is a curve C with only nodes and such that every irreducible component is rational because one can smooth a certain set of nodes in order to obtain an irreducible curve with g nodes and then one can smooth any subset consisting of $g - t$ nodes.

First assume $m_4 > 0$. Consider the strict trasform Z in S of a conic through P_1, P_2, P_3, P_4 but not through P_5, P_6. Then $Z^2 = 0$ and $L \otimes \mathcal{O}(-Z) \simeq \simeq (a-2, m_1', \ldots, m_6')$ has again, up to permutation, this as adequate basis except in one case:

a) $m_1 = \cdots = m_6$, $3 m_1 = a$, in which case $m_1 = m_1' = m_2'$, $m_i' = m_i - -1$, $i > 2$.

If *a*) holds we choose as Z the strict transform of a conic through P_1, P_2, P_3, P_4, P_5. Then the given basis is adequate for $L \otimes \mathcal{O}(-Z)$ and the condition $m_1 + \cdots + m_6 = 2a$ cannot hold. Thus we can repeat the construction (case *a*) no longer occur), choosing 4 points P_i's such that the corresponding m_i's are maximal, until $m_4' = 0$.

Now assume $m_4 = 0$. Since the basis is adequate, we can choose m_i strict transforms of general line through P_i, $i = 1, 2, 3$, and $a - m_1 - m_2 - m_3$ pull-back of lines in general position, obtaining a reducible curve C in | D | with rational irreducible components. Since all the irreducible components, except possibly the component considered in case *a*), vary in base point free linear systems, we may assume that C has only nodes as singularities and we are through. Q.E.D.

§ 3. THE QUARTIC SURFACE

Next we shall show that on a particular type of quartic surface in $\mathbf{P}_3$ with double line, for every d, g, t with $0 \leq t \leq g \leq (d-1)^2/8$ there exist irreducible curves of degree d, arithmetic genus g and t nodes. We modify slightly Gruson-Peskine's construction because we were unable to prove this result on the quartic surface image of $\mathbf{P}_2$ blown-up at 9 general points.

Instead we use the blow-up of $\mathbf{P}_2$ at 9 points lying on a nodal cubic curve.

Let E be an irreducible cubic curve in $\mathbf{P}_2$ with a node T. Let $P_1, \ldots, P_9$ be 9 points ($\neq$ T) on E and let $p : S \to \mathbf{P}_2$ be the blow-up of $\mathbf{P}_2$ at the 9 points P_i's. Pic (S) has as basis the pull-back of a line l and $-l_i$, where l_i is the exceptional line $p^{-1}(P_i)$. In this basis a divisor is written as $(a, m_1, \ldots, m_9)$. The canonical divisor is $(-3, -1, \ldots, -1)$. This basis gives an isomorphism Pic (S) $\simeq \mathbf{Z}^{10}$ and then we have on $\mathbf{Z}^{10}$ a symmetric non-degenerate form (,) induced by the intersection form.

The system $-\omega_s$ contains a divisor C, the strict transform of the cubic E. We assume the following condition to hold true.

i) The natural map $\mathrm{Pic}(S) \to \mathrm{Pic}(C)$ is injective.

Condition *i*) can be achieved if the P_i's are general points of E because $\mathrm{Pic}^0(C) \cong \mathbf{C}^*$. In particular condition *i*) implies that no 3 points P_i's are collinear, no 6 P_i's are on a conic and that there is only one cubic, E, passing through $P_1, \ldots, P_9$.

Consider the group G of automorphisms of Pic (S) which preserve the intersection form and the canonical divisor. For every $g \in G$, there exists a morphism $v \colon S \to \mathbf{P}_2$, blowing up of 9 points Q_i on $v(C)$ such that $g(1) = v^*(\mathcal{O}_{P2}(1))$ and $g(1_i) = v^{-1}(Q_i)$; indeed the proof of [5] prop. 1.2 works verbatim. Fix a representation p of S as a blow-up of the plane and put $c = (1, 1, 0, \ldots, 0)$ and $b = (0, 0, \ldots, 0, -1, -1)$. Thus c is the strict transform of a line in $\mathbf{P}_2$ passing through P_1 but not through P_j for $j \neq 1$, and b is the disjoint union of the two exceptional curves l_8 and l_9.

We have ([5] Lemme 1.5) $h^0(n\omega_s) = 1$ for $n < 0$, $h^0(n\omega_s) = 0$ for $n > 0$ and $h^1(n\omega_s) = 0$ for $n \in \mathbf{Z}$ because *i*) implies that the normal bundle of C in S is not a torsion point of Pic (C). Furthermore $|c - \omega_s|$ is without base points and gives a bi-rational morphism $u \colon S \to \mathbf{P}_3$ where $u(S)$ is a quartic surface with double line $u(C)$; u is bi-regular outside C and is $2-1$ on C; for every P in C, $u^{-1}u(P)$ is the variety of zeros of the unique section of $(c - \omega_s)$ vanishing at P ([5], Lemma 1.6). Again as in [5], Prop. 1.7 and Corollary 1.8 we have, using Riemann-Roch and a vanishing theorem for divisors on an irreducible, reduced curve (see [3], pag. 59) 1) the divisor $c - n\omega_s$ is generated by global sections for $n > 0$, $b - n\omega_s$ is generated by global sections for $n > 1$, $h^1(b - n\omega_s) = 0$ for $n \geq 0$, $h^1(c - n\omega_s) = 0$ for $n \geq -1$.

2) A general section of $|c - n\omega_s|$, $n \geq 0$, (resp. $|b - n\omega_s|$, $n \geq 1$) vanishes on a connected smooth curve of genus $2n$ (resp. $2n - 1$).

The key point in Gruson-Peskine's proof is their Prop. 1.9 stating that $(c, g(c))$ and $(b, g(c))$ go through all the integers ≥ 0 as g varies in G. But we are lucky since their proof uses only properties of the intersection form of S, which is identical to those of the surface they used. First they prove the existence on S of smooth connected curves of genus g and degree d for $0 \leq g \leq d - 3$. Here the degree of a curve H is $(H, c - \omega_s)$ i.e. the degree of its image $u(H)$ in $\mathbf{P}_3$ if H does not contain C as an irreducible component. Gruson and Peskine found such a curve in $|\sigma(c - k\omega_s)|$ if $g = 2k$ (and $\sigma \in G$ is such that $(c, \sigma(c)) = d - g - 2$), in $|\sigma(b - k\omega_s)|$ if $g = 2k - 1$ (then $\sigma \in G$ is such that $(\sigma(b), c) = d - g - 3$).

If $d - 3 < g \leq (d-1)^2/8$, set $d(r) = d - 4r$, $g(r) = g + r(2r - d - 1)$.

Then there exists an integer r such that $0 \leq g(r) \leq d(r) - 3$. Gruson and Peskine found a connected non-singular curve of genus g and degree d in the system $|w + r(c - \omega_s)|$ where $|w|$ is the system in which they found a non-singular connected curve of genus $g(r)$ and degree $d(r)$. Their proofs work verbatim for our surface S. Our strategy is simple:

1) find an irreducible rational curve with g nodes A, in the same linear system in which they found a smooth curve of genus g;

2) show that for a good choice of A, u maps isomorphically A to u (A). Thus u (A) is an irreducible curve of degree d and arithmetic genus g in $\mathbf{P}_3$ with exactly g nodes.

3) Play the same game as in 1) and 2) for $t < g$ nodes.

Step 2) consists in finding such a curve A which does not pass through the node of C, intersects transversally C and is such that $C \cap A$ does not contain any pair of points with the same image in $\mathbf{P}_3$.

LEMMA 1. *Let* F *be a rational surface,* $|D|$ *a linear system with* D *smooth and connected and of genus* g, $D \cdot \omega_F \leq -2$. *Let* H *be an irreducible family of curves* $A \in |D|$ *with* g *nodes,* $\dim H = \dim |D| - g$. *Then not all the curves in* H *pass through the same point* P *of* F.

Proof. Assume that every $A \in H$ contains a point P and blow-up P. First assume that the general $A \in H$ does not have a node at P; restricting H we may assume that no $A \in H$ has a node at P. Let F′ be the surface obtained blowing-up F at P and let H′ the family of the strict transforms A′ of the curves $A \in H$; $\dim H' = \dim H$, $A' \cdot A' = A \cdot A - 1$, $A' \cdot \omega_{F'} = A \cdot \omega_F - 1$ and Riemann-Roch implies $\chi(F', \mathcal{O}(A')) = \chi(F, \mathcal{O}(A)) - 1$. Since F′ is rational, $H^2(F', \mathcal{O}(A')) = 0$; from the exact sequence on F′

$$0 \to \mathcal{O} \to \mathcal{O}(A') \to \mathcal{O}(A')_{|A'} \to 0 .$$

Since $A' \cdot \omega_{F'} \leq -1$, we obtain $h^1(\mathcal{O}(A')) = 0$ since $\mathcal{O}(A')_{|A'}$ is a line bundle of degree $> 2g - 2$ on an irreducible curve of arithmetic genus g (see [3], pag. 59).

Thus $h^0(F', \mathcal{O}(A')) = h^0(F, \mathcal{O}(A)) - 1$, $\dim H' = \dim H$, contradicting the independence of nodes on a rational surface ([8], 2.2 and 2.13).

If at the base point P every $A \in H$ has a node, A′ has arithmetic genus $g - 1$. $A' \cdot A' = A \cdot A - 4$, $A' \cdot \omega_{F'} = A \cdot \omega_F + 2 \leq 0$, $h^1(F', \mathcal{O}(A')) \leq 1$, $\chi(F', \mathcal{O}(F')) = \chi(F, \mathcal{O}(A)) - 3$, $h^0(F', \mathcal{O}(A')) \leq h^0(F, \mathcal{O}(A)) - 2$ and, since A′ has $g - 1$ nodes, again we have a contradiction. Q.E.D.

LEMMA 2. *There exists in* $|\sigma(c - k\omega_s)|$ (*resp.* $|\sigma(b - k\omega_s)|$), $k \geq 1$, *an irreducible family* H'_k (*resp.* H_k) *of irreducible rational curves with* $2k$ (*resp.* $2k - 1$) *nodes and every such maximal irreducible family has dimension* $h^0((c - k\omega_s)) - 2k$ (*resp.* $h^0((b - k\omega_s)) - 2k + 1$). *Furthermore, for a general curve* D *in* H'_k *or* H_k, D *intersects* C *transversally at* 2 *different non-conjugate points, except for* H'_1 *if* σ *is the identity of* G.

Proof. The last part follows from $(\sigma(c), C) = (\sigma(b), C) = 2$ and Lemma 1. We prove the first part by induction on k; we consider only the case H_k since the other case is similar.

We woık by induction on k; since every $\sigma \in G$ induces an isomorphism of S, we may assume $\sigma = \mathrm{id}$. We want to prove that the family of irreducible, rational curves in $|b - \omega_s|$ which are deformation of $l_8 \cup l_9 \cup C$ with $l_8 \cap C$ and $l_9 \cap C$ as unassigned nodes is irreducible, smooth and of dimension 1. To prove this we cannot apply the statement of [8], Lemma 2.2; we prove directly the vanishing of $H^1(\hat{N}_1)$ which gives the result by Wahl's theory [1], Lemma 9.60.

Here we explain the notation: $\hat{N}_1$ is the normal bundle to $D := l_8 \cup l_9 \cup C$ in S tensored with the ideal sheaf of the node T of C.

Let $n : D' \to D$ the partial normalization of D at the point T; put $C' = n^{-1}(C)$.

We have $\hat{N}_1 = n_* (n^* (\mathcal{O}_D(D)) \otimes \mathcal{O}_{D'}(-p-q))$, where $\{p, q\} = n^{-1}(T)$ ([1], 9.59).

Then, since n is affine, it is sufficient to prove $H^1(D', R) = 0$, where $R := n^* (\mathcal{O}_D(D)) \otimes \mathcal{O}_{D'}, (-p-q)$. Note that R is locally free.

We have the Mayer-Vietoris exact sequence

$$0 \to R \to R_{|l_8 \cup l_9} \oplus R_{|C'} \xrightarrow{u} R_{|(l_8 \cup l_9) \cap C'} \to 0 \tag{1}$$

and $h^1(R_{|l_8 \cup l_9}) = h^1(R_{|C'}) = 0$ since, for example, $\deg(R_{|C'}) > -2$ and C' is rational and smooth. Furthermore $R_{|l_8 \cup l_9}$ is trivial and thus u_* is surjective, since l_8 and l_9 are disjoint.

The inductive step is similar. By the last part of the lemma, $D := E \cup C$ has only nodes as singularities, where E is a general element of H_k. We take as H_{k+1} the family obtained from $E \cup C$, E general in H_k, killing one of the nodes in $E \cap C$. Let D' be the partial normalization of $D = E \cup C$ at all the singular points of $E \cup C$ except one in $E \cap C$. The we apply again the Mayer-Vietoris exact sequence (1). Q.E.D.

Lemma 2 gives immediately, by Gruson-Peskine's proof, the existence of nodal curves of degree d and arithmetic genus g if $d \leq g + 3$.

Now assume $d > g + 3$. Put $g(r) = g - r(2r - d - 1)$, $d(r) = d - 4r$ and take r such that $0 \leq g(r) \leq d(r) - 3$. Let $U = \sigma(c - k\omega_s)$ or $\sigma(b - k\omega_s)$ be the linear system in which we found a family H of rational nodal curves of degree $d(r)$ and genus $g(r)$. We can work by induction on r. The only problem is to be sure that at each step we obtain something X with good intersection with C (i.e. transversal and no conjugate points) and such that we can add a rational curve D in the expected linear system and such that $X \cup D$ has only nodes and good intersection with C. To a general member $A \in H$, first we add C and then, using the proof of Lemma 2, we kill one node in $A \cap C$ obtaining a rational curve A' with only nodes and with good intersection with C; the last part follows from Lemma 1 as at the beginning of the proof of Lemma 2, since $(A' \cdot C) = 2$.

Then we repeat the construction adding (r — 1)-times C, killing a node at each step. We obtain a rational curve Y with only nodes and good intersection with C. Then we add r strict transforms of general lines through P_1, say $A_1, \ldots, A_r$. For a general choice of the A_i''s, $(A_1' \cup \ldots \cup A_r') \cap C$ does not contain conjugate points by condition i) and does not contain points conjugate to points in $Y \cap C$, for a general choice, since they are finite. Furthermore we may assume that $Y \cup A_1' \cup \ldots \cup A_s'$ has only nodes; in fact for every irreducible curve Z in the plane, a general line through P_1 intersects Z transversally outside P_1 and at P_1 it is not in the tangent cone to Z at P_1, by the bi-duality of the Gauss map (we are in characteristic 0).

Thus the proof of the theorem is complete.

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