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Lie group structures and reproducing kernels on the unit ball of $\mathbb{C}^n$

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Geometria. — *Lie group structures and reproducing kernels on the unit ball of* $\mathbf{C}^n$. Nota di UMBERTO SAMPIERI (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si introducono due strutture di gruppo di Lie su un dominio di Siegel omogeneo di $\mathbf{C}^n$. Per la palla unitaria si definisce una famiglia ad un parametro di strutture intermedie; ad ognuna di esse viene associato naturalmente un nucleo riproducente ottenendo un'interpolazione tra il nucleo di Bergman ed il nucleo di Szego.

INTRODUCTION

It is well known that, denoting respectively by K_1 and K_0 the Bergman and the Szego kernels of a symmetric domain D, there exists a real number $0 < a_0 < 1$ such that $K_0 = (K_1)^{\alpha_0}$.

Consequently it is natural to ask for which real number α there exists a Hilbert space of holomorphic functions on D, whose reproducing kernel is given by $(K_1)^a$. A complete answer to this question has been given by Vergne and Rossi (|3|), showing the link between representations of semi-simple Lie groups and the theory of reproducing kernels.

In this brief note we announce some results on Lie group structures on homogeneous Siegel domains, focusing our attention on a simple but not trivial example: the unbounded realization of the unit ball in $\mathbf{C}^n$.

The introduction of these Lie group structures improves the understanding of function theory on such domains and is summarized in section 2.

In section 1 we fix notations and recall a few facts about Siegel domains (see |2| for proofs and details) while section 3 is completely devoted to the discussion of the example. Proofs and general results will appear in a forthcoming paper (|5|).

1. AFFINELY HOMOGENEOUS SIEGEL DOMAINS

Let V be an homogeneous sharp cone in $\mathbf{R}^n$. We shall denote by Aut (V) the group of all linear transformations of $\mathbf{R}^n$ preserving V. Let $Q : \mathbf{C}^m \times \mathbf{C}^m \rightarrow \mathbf{C}^n$ be a V-hermitian-homogeneous form that is an hermitian form such that:

i) $Q(u, u)$ belongs to V for each u in $\mathbf{C}^m$, $u \neq 0$;

ii) for each A belonging to Aut (V) thre exists a B in $Gl(m, \mathbf{C})$ such that $AQ(u, u) = Q(Bu, Bu)$ for each u in $\mathbf{C}^m$.

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With these data we can construct the affinely homogeneus domain in $\mathbf{C}^{n+m}$:

$$\text{(1.1)} \qquad \mathrm{D}(\mathrm{V}, \mathrm{Q}) = \{w = (z, u) \in \mathbf{C}^{n+m} : c(w) = \mathrm{Im}(z) - \mathrm{Q}(u, u) \in \mathrm{V}\} .$$

The Silov boundary of D (V , Q) is given by:

$$\text{(1.2)} \qquad \mathrm{S}(\mathrm{Q}) = \{w \in \mathbf{C}^{n+m} : c(w) = 0\} .$$

The group N (Q) of affine invertible transformations of $\mathbf{C}^{n+m}$

$$\text{(1.3)} \qquad \mathrm{N}(\mathrm{Q}) = \{g = g(x_0, u_0), x_0 \in \mathbf{R}^n, u_0 \in \mathbf{C}^m\}$$

where

$$g(x_0, u_0)(z, u) = (z + x_0 + i(2\,\mathrm{Q}(u, u_0) + \mathrm{Q}(u_0, u_0)), u + u_0)$$

preserves D (V , Q) and acts simply transitively on S (Q). It is well known that N (Q) is a two-step nilpotent Lie group.

2. Two Lie group structures on a homogeneous Siegel domain

In | 4 | Vinberg has shown that there exists a triangular subgroup G (V) of Aut (V) acting in a simply transitive way on V. Therefore if we select a point e in V for each $x \in \mathrm{V}$ there exists a unique A (x) in G (V) such that $\mathrm{A}(x)\, e = x$. The mapping $x \rightarrow \mathrm{A}(x)$ turns out to be a C^∞ diffeomorphism of V onto G (V). Consequently we can define a product on V by setting:

$$\text{(2.1)} \qquad x \cdot y = \mathrm{A}(x)\, y \qquad \text{for each } x, y \text{ belonging to V} .$$

Clearly e is the unit element of the Lie group $(\mathrm{V}, \cdot)$.

Let us consider the two subsets of D (V , Q):

$$\text{(2.2)} \qquad \tilde{\mathrm{V}} = (ix, 0) \quad , \quad x \in \mathrm{V}$$

$$\text{(2.3)} \qquad \mathrm{S}(\mathrm{Q}, e) = \{w \in \mathrm{D}(\mathrm{V}, \mathrm{Q}) : c(w) = e\}$$

$\tilde{\mathrm{V}}$ is a group, homomorphic to $(\mathrm{V}, \cdot)$, with the product:

$$\text{(2.4)} \qquad (ix, 0)\, \mathrm{X}_0\, (iy, 0) = (i(x \cdot y), 0) .$$

We define a group structure on S (Q , e), homomorphic to N (Q), setting:

$$\text{(2.5)} \qquad (z, u)\, \mathrm{X}_0\, (z', u') = (z + z' + i(2\,\mathrm{Q}(u', u) - e), u + u') .$$

Since $S(Q, e) \cap \tilde{V} = \{(ie, 0)\}$ and the mapping $P : D(V, Q) \to \tilde{V} \times S(Q, e)$ defined by $P(w) = P(z, u) = ((ic(w), 0), (\mathrm{Re}(z) + ie + iQ(u, u), u))$ is a diffeomorphism, we can introduce a direct product structure on $D(V, Q)$. A straightforward calculation shows that the product of two generic elements is given by:

$$w X_0 w' = (z, u) X_0 (z', u') = \tag{2.6}$$

$$= (z + z' + i(c(w) \cdot c(w')) - ic(w) - ic(w') + 2iQ(u', u), u + u').$$

The mapping $c : (D(V, Q), X_0) \to (V, \cdot)$ is a surjective homomorphism. In order to define another Lie group structure on $D(V, Q)$ such that left translations are holomorphic automorphisms, we have to look for a semidirect product of V and $S(Q, e)$.

That is obtained by means of the following lemma ([5]):

LEMMA. *There exists a group homomorphism* $s : G(V) \to Gl(m, \mathbf{C})$ *such that*:

$$AQ(u, u) = Q(s(A)u, s(A)u) \qquad \textit{for each } A \textit{ in } G(V), \textit{ for each } u \textit{ in } \mathbf{C}^m. \tag{2.7}$$

In fact, given s, we can define the requested product by setting:

$$wX_1 w' = (z, u) X_1 (z', u') = \tag{2.8}$$

$$= (A(c(w)) z' + z - ic(w) + 2iQ(s(A(c(w))) u', u), s(A(c(w))) u' + u).$$

Clearly $(\tilde{V}, X_0)$ and $(S(Q, e), X_0)$ are subgroups of $(D(V, Q), X_1)$ and c is still a surjective homomorphism.

It is important to observe that, denoting by K_1 the Bergman Kernel of $D(V, Q)$, normalized by the condition $K_1((ie, 0), (ie, 0)) = 1$:

i) $\omega_1 : (D(V, Q), X_1) \to \mathbf{R}^+$, $\omega_1(w) = K_1(w, w)^{-1}$ is a surjective homomorphism of Lie groups;

ii) $d\mu_1(w) = K_1(w, w)\, dm(w)$ is a left invariant Haar measure on $(D(V, Q), X_1)$;

iii) $\to \partial \bar{\partial} \ln \omega_1$ defines a positive definite hermitian left invariant complete metric on $(D(V, Q), X_1)$;

iv) the reproducing formula for square summable holomorphic functions on $D(V, Q)$ can be expressed as a convolution on $(D(V, Q), X_1)$, namely:

$$F(w') = \int L(w^{-1} X_1 w') F(w)\, d\mu_1(w) \quad , \quad L(w) = K_1(w, (ie, 0)). \tag{2.9}$$

The following question then arises: are there « in between » Lie group structures on D (V , Q) and how are they related to Hilbert spaces of holomorphic functions on D (V , Q)? We shall clarify and answer this question for the special case of the unit ball in the next section.

3. The unit ball in $\mathbf{C}^{n+1}$

We shall denote points w in $\mathbf{C}^{n+1}$ by $w = (z , u)$, z in $\mathbf{C}$, $u = (u_1 , \ldots , u_n)$ in C^n and consider the Siegel domain:

$$\mathrm{D}_{n+1} = \{w \text{ in } \mathbf{C}^{n+1} : c(w) = \mathrm{Im}(z) = \langle u , u \rangle \in \mathbf{R}^+\} \tag{3.1}$$

where $\langle\, , \rangle$ is the ordinary hermitian product in $\mathbf{C}^n$. It is well known that D_{n+1} is biholomorphically equivalent to the unit ball in $\mathbf{C}^{n+1}$.

Selecting $(i , 0)$ as unit element, the normalized Bergman and Szego kernels for D_{n+1} are uniquely characterized by the following identities:

$$\mathrm{K}_1 (w , w) = c(w)^{-n-2} \tag{3.2}$$

$$\mathrm{K}_0 (w , w) = c(w)^{-n-1} = \mathrm{K}_1 (w , w)^{a_0} , \quad a_0 = (n+1)/(n+2). \tag{3.3}$$

According to general results summarized in section 2 we can define two Lie group structures on D_{n+1}, namely:

$$w \, \mathrm{X}_0 \, w' = (z + z' - ic(w) - ic(w') + ic(w)\, c(w') + 2i\langle u' , u \rangle ,\ u + u') \tag{3.4}$$

$$w \, \mathrm{X}_1 \, w' = (c(w)\, z' - ic(w) + z + 2i\sqrt{c(w)}\,\langle u' , u \rangle ,\ \sqrt{c(w)}\, u' + u) \tag{3.5}$$

S (i) is a subgroup for both the structures, homomorphic to the Heisemberg group.

Let us consider the one parameter family of diffeomorphisms $\mathrm{C}_h : \mathrm{D}_{n+1} \to \mathrm{D}_{n+1}$, h in $\mathbf{R}^+$ defined by:

$$\mathrm{C}_h (w) = (ic(w)^h + i\langle u , u \rangle + \mathrm{Re}(z) , u) . \tag{3.6}$$

The following identities are easy to check:

$$\begin{cases} \mathrm{C}_h \, \mathrm{C}_k = \mathrm{C}_{hk} & \text{for each } h , k \text{ in } \mathbf{R}^+ \\ \mathrm{C}_1 = id & \\ \mathrm{C}_{h|\mathrm{S}(i)} = id_{|\mathrm{S}(i)} & \text{for each } h \text{ in } \mathbf{R}^+ \\ \lim\limits_{h \to 0} \mathrm{C}_h (w) & \text{belongs to S}\,(i) \text{ for each } w \text{ in } \mathrm{D}_{n+1}. \end{cases} \tag{3.7}$$

We shall say that $\{C_h\}_{h\in\mathbf{R}^+}$ is a semigroup of deformations of D_{n+1} to S (i). We define a family of products X_h on D_{n+1} by setting:

$$w \, X_h \, w' = C_{1/h} \left(C_h(w) \, X_1 \, C_h(w') \right). \tag{3.8}$$

A straightforward calculation shows that:

$$\begin{aligned} &w \, X_h \, w' = \\ &= (c(w)^h z' - i\,c(w) + z + 2i\sqrt{c(w)^h} < u', u > + \, i\,c(w')\,(c(w) - \\ &\quad - c(w)^h), \sqrt{c(w)^h}\, u' + u). \end{aligned} \tag{3.9}$$

Therefore we have:

$$\lim_{h\to 0} w \, X_h \, w' = w \, X_0 \, w' \tag{3.10}$$

and this accounts for the expression « in between » structures previously used.

A left invariant Haar measure on (D_{n+1}, X_h) is given by:

$$d\mu_h(w) = K_0(w, w)^{h-1} \, d\mu_1(w) \tag{3.11}$$

and the mappings $\omega_h : (D_{n+1}, X_h) \to \mathbf{R}^+$

$$\omega_h(w) = K_1(C_h(w), C_h(W))^{-1} \tag{3.12}$$

are surjective Lie groups homorphisms.

We can introduce the Bergman weighted spaces:

(3.13) $F_h(D_{n+1}) = \{F$ holomorphic functions on D_{n+1} such that:

$$\|F\|_h = \left(\int |F(w)|^2 \, \omega_h(w) \, d\mu_h(w) \right)^{1/2} < \infty \}. \tag{3.14}$$

The following theorem holds: (| 5 |)

THEOREM

i) $F_h(D_{n+1}) \neq 0$ *for each* h *in* $\mathbf{R}^+$;

ii) $F_1(D_{n+1})$ *it the ordinary Bergman space*;

iii) *If we denote gy* K_h *the normalized reproducing kernels of* $F_h(D_{n+1})$ *we have*:

a) $K_h = (K_0)^{1+h/n+1}$

b) $K_h(\cdot, w)$ *belongs to the Hardy space* $H^2(D_{n+1})$ *for each* w *in* D_{n+1}, *for each* h *in* $\mathbf{R}^+$

c) $\lim_{h \to 0} K_h(\cdot, w) = K_0(\cdot, w)$ *for each* w *in* D_{n+1}, *where the limit is taken with respect to the topology of* $H^2(D_{n+1})$.

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