Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **76** (1984), n.2, p. 85–87.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1984_8_76_2_85_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1984.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta dell'11 febbraio 1984

Presiede il Presidente della Classe GIUSEPPE MONTALENTI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Algebra. — *An application of commutator theory to incidence algebras.* Nota di PAOLO LIPPARINI (*) presentata (**) dal Socio G. ZAPPA.

RIASSUNTO. — Usando la teoria del commutatore in algebra universale, si dimostra che una larga classe di algebre di incidenza sono polinomialmente equivalenti a moduli su anelli con divisione.

Commutator theory in Universal Algebra has been first developed in [8] for the case of permutable varieties; then it was extended in [5] to the more general class of modular varieties. The commutator is a binary operation on congruences; in groups it corresponds to the usual commutator of two normal subgroups, in commutative rings to the product of two ideals.

In this paper we give some applications of commutator theory to incidence algebras: $\mathcal{U}$ is called an *incidence algebra* if it is non-simple and every non-zero principal congruence of $\mathcal{U}$ is an atom of Con $\mathcal{U}$. The name comes from the fact that this happens if the geometry of congruence classes $\Gamma(\mathcal{U})$ is an incidence geometry (see [7], in particular Theorem 1.7).

Typical incidence algebras are modules over a division ring; we prove that, except for some pathological cases, all incidence algebras have essentially this form.

Notations and terminology are standard, and can be found in the textbooks [1], [2] and [3]. The following definition is suggested by [4, 2.12] and [9, Lemma 7]; notice that we do not assume that $\mathcal{U}$ belongs to a modular variety.

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DEFINITION. *Given* α, β *congruences of an algegra* $\mathcal{U}$, *the* commutator of α and β, *in symbols* $[\alpha, \beta]$, *is the least congruence* γ *of* $\mathcal{U}$ *such that, for every* $\bar{x}$, $\bar{y} \in A^n$, p, $q \in A$, *with* $p \beta q$ *and* $x_i \alpha y_i$ $(0 \leq i < n)$, *and for every* $n+1$-*ary term* t:

$$t(\bar{x}, p) \gamma t(\bar{y}, p) \quad \textit{implies} \quad t(\bar{x}, q) \gamma t(\bar{y}, q).$$

Facts. (i) $[\beta, \bigvee_{j \in J} \alpha_j] = 0$ *iff* $\bigvee_{j \in J} [\beta, \alpha_j] = 0$;

(ii) $[\alpha, \beta] \subseteq \alpha \wedge \beta$;

(iii) $[\alpha, \beta] = 0$ *iff for every two finitely generated congruences* $\alpha' \subseteq \alpha$ *and* $\beta' \subseteq \beta$, *it happens that* $[\alpha', \beta'] = 0$;

(iv) $[\cdot, \cdot]$ *is monotone in both arguments.*

PROPOSITION. *If* $\alpha \neq 1_{\mathcal{U}}$ *is a congruence of an incidence algebra* $\mathcal{U}$, *then* $[\alpha, 1_{\mathcal{U}}] = 0$. *Moreover,* $[1_{\mathcal{U}}, 1_{\mathcal{U}}] = 0$, *if* $1_{\mathcal{U}}$ *is not finitely generated.*

Proof. If $a, b \in A$ and $(a, b) \notin \alpha$, then $[\alpha, \mathrm{Con}(a, b)] \subseteq \alpha \cap \mathrm{Con}(a, b) = 0$, because $\mathcal{U}$ is an incidence algebra; moreover, since $\alpha \neq 1_{\mathcal{U}}$, $1_{\mathcal{U}} = \bigvee \{\mathrm{Con}(a, b) \mid (a, b) \notin \alpha\}$, and the first part follows from (i). Now the second part is an immediate consequence of (iii). —|

From the arguments in the proof of [9, Theorem 2] it follows:

LEMMA. *If* $\mathcal{U}$ *belongs to a k-permutable variety and* $[\alpha, \alpha] = 0$, *for every principal congruence* α *of* $\mathcal{U}$, *then* $\mathcal{V}(\mathcal{U})$ *is permutable (hence, modular).*

THEOREM 1. *If* $\mathcal{U} \in \mathcal{V}$ *is an incidence algebra and* $\mathcal{V}$ *is a modular or a k-permutable variety, then* $\mathcal{U}$ *is polynomially equivalent to a module over a division ring.*

Proof. In modular varieties the commutator is distributive over joins [5, Theorem 2.2], hence $[1_{\mathcal{U}}, 1_{\mathcal{U}}] = \bigvee \{[\alpha, 1_{\mathcal{U}}] \mid \alpha \text{ is principal}\} = 0$, because of the Proposition. It follows from [9, Cor. 2] that $\mathcal{U}$ is polynomially equivalent to a module over some ring, which is shown to be a division ring by standard arguments, using the fact that $\mathcal{U}$ is an incidence algebra.

If $\mathcal{V}$ is k-permutable, the Lemma, the Proposition and (iv) above imply that $\mathcal{V}(\mathcal{U})$ is modular, and we fall in the preceding case. —|

Theorem 1 says that an incidence algebra $\mathcal{U}$ is essentially a module over a division ring, provided that $\mathcal{U}$ generates either a modular variety or a k-permutable variety. Nevertheless, we can still say something even if we know nothing about the variety generated by $\mathcal{U}$.

THEOREM 2. *If $\mathscr{U}$ is an incidence algebra with permutable congruences, and either (a) Con $\mathscr{U}$ has height at least 3, or (b) in $\mathscr{U}$ some non-zero principal congruence has a finite block, then $\mathscr{U}$ has an expansion polynomially equivalent to a module over a division ring.*

Proof. Because of [6, Prop. C] the congruence class geometry $\Gamma(\mathscr{U})$ is affine; we prove that it is also arguesian: if (*a*) holds, the dimension of $\Gamma(\mathscr{U})$ is greater than 2, and the arguesianity follows from a well known geometrical result. If, on the contrary, $\Gamma(\mathscr{U})$ has dimension 2, but (*b*) holds, then $\Gamma(\mathscr{U})$ is a finite affine plane, which is arguesian because of [6, Lemma 5].

So, we may add a new ternary term t to $\mathscr{U}$, in such a way that $t(a, b, c) = d$ iff $\vec{bd} = \vec{ba} + \vec{bc}$ (in a coordinatization of $\Gamma(\mathscr{U})$).

If $\mathscr{U}^+$ is the expanded structure, the congruences of $\mathscr{U}$ and $\mathscr{U}^+$ are the same; and $\mathscr{U}^+$ generates a permutable variety, as t satisfies the Mal'cev identities. Since permutable varieties are modular, the hypothesis of Theorem 1 is satisfied, and $\mathscr{U}^+$ is an expansion of $\mathscr{U}$ polynomially equivalent to a module over a division ring. —|

It is an open problem whether $[1_{\mathscr{U}}, 1_{\mathscr{U}}] = 0$ holds for every incidence algebra $\mathscr{U}$.

BIBLIOGRAPHY

[1] S. BURRIS and H.P. SANKAPPANAVAR (1981) – *A course in Universal Algebra*, GTM 78, New York.
[2] P.M. COHN (1965) – *Universal Algebra*, New York.
[3] G. GRATZER (1979²) – *Universal Algebra*, New York.
[4] H.P. GUMM (1980) – *An easy way to the commutator in modular varieties*, « Arch. Math. », *34*, 220-228.
[5] J. HAGEMANN and C. HERRMANN (1979) – *A concrete ideal multiplication for algebraic systems and its relation to congruence distributivity*, « Arch. Math. », *32*, 234-245.
[6] A. PASINI (1980) – *On the finite transitive incidence algebras*, « Boll. U.M.I. », *17-B*, 373-389.
[7] A. PASINI (1977) – *On the incidence algebras*, « Le Matematiche », *XXXI*, *I*, pp. 138-147.
[8] J.D.H. SMITH (1976) – *Mal'cev Varieties*, LNM 554, Berlin.
[9] W. TAYLOR (1982) – *Some applications of the term condition*, « Alg. Univ. », *14*, 11-24.