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Fausto Sacerdote, Fernando Sansò

The current situation in the linear problem of Molodenskii

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Geodesia. — *The current situation in the linear problem of Molodenskii.* Nota II di FAUSTO SACERDOTE e FERNANDO SANSÒ, presentata (*) dal Socio L. SOLAINI.

RIASSUNTO. — Si prova l'esistenza di un'unica soluzione debole che dipende con continuità dai dati al contorno per il problema lineare di Molodenskii in approssimazione quasi sferica, nel caso che la superficie al contorno soddisfi una condizione di cono.

Si segue un approccio costruttivo diretto, che generalizza una procedura precedentemente elaborata per il problema semplice di Molodenskii.

Inoltre si prova che la soluzione ha derivate prime a quadrato integrabile al contorno, il che è essenziale per le applicazioni geodetiche.

1. INTRODUCTION

In the preceding Note 1 [5] the existence and uniqueness of a weak solution of linear Molodenskii's problem in almost spherical approximation

$$(1.1)\quad \begin{cases} \Delta \mathrm{T} = 0 \quad \text{in} \quad \Omega \\ -\boldsymbol{m}_0 \cdot \nabla \mathrm{T} + \mathrm{T}|_{\partial\Omega} = u + \boldsymbol{a} \cdot \mathbf{A} \quad \left(u \in \mathrm{H}^{1/2}(\partial\Omega)\,,\ \mathrm{A}_j = \left(\dfrac{\mathrm{R}}{r}\right)^2 \mathrm{Y}_{1j}|_{\partial\Omega}\right) \\ \mathrm{T} = \dfrac{\alpha}{r} + 0\,(r^{-3}) \qquad \left(\boldsymbol{m}_0 \text{ close to } -\dfrac{1}{2}\,r\right) \end{cases}$$

was found for $\Omega \in \mathcal{N}^{(1),1}$; yet the method used there fails if we assume $\Omega \in \mathcal{N}^{(0),1}$, a more natural condition in geodesy. In the present note we adopt a different approach, that is a generalization of the direct method used by Sansò [8] for simple Molodenskii's problem. This method consists of extending the boundary condition $\dfrac{1}{2}\, r\, \dfrac{\partial \mathrm{T}}{\partial r} + \mathrm{T}\,|_{\partial\Omega} = u + \boldsymbol{a} \cdot \mathbf{A}$ over the whole Ω, taking advantage of the fact that, if T is harmonic, so is $\dfrac{1}{2}\, r\, \dfrac{\partial \mathrm{T}}{\partial r} + \mathrm{T}$. Moreover it is easy to obtain T from u outside a sphere, by means of a development into spherical harmonics; thus the main problem is the construction of T in the portion of space between $\partial\Omega$ and a sphere that encloses it. The result is obtained by a line integration along a radius, that is allowed owing to the regularity of the harmonic function u.

(*) Nella seduta del 14 gennaio 1984.

In the general case, $\boldsymbol{m}_0 \neq -\frac{1}{2}\boldsymbol{r}$, the difficulty arises because the harmonic character of T does not imply that $-\boldsymbol{m}_0 \cdot \nabla T + T$ is harmonic too. On the other hand, we are not forced to extend the boundary condition over Ω by using the isozenithal field $\boldsymbol{m}_0$ in the whole domain Ω. It is more convenient to define a new C^1 vector field $\boldsymbol{m}$ that agrees with $\boldsymbol{m}_0$ on $\partial\Omega$ and is equal to $-\frac{1}{2}\boldsymbol{r}$ in the domain Ω_R outside a spherical surface S_R with suitable radius R.

To this propose, let $S_{R'}$ be a spherical surface enclosing S, with $R' < R$ (fig. 1); let $\Phi : [0, 1] \to \mathbf{R}$ be a C^∞ function with $\Phi(0) = 1$, $\Phi'(0) = \Phi'(1) = \Phi(1) = 0$.

Then $\boldsymbol{m}$ can be defined in the following way:

$$
\begin{aligned}
&\boldsymbol{m} = \boldsymbol{m}_0 = -\frac{1}{2}\boldsymbol{r} + \mu_0 \qquad \text{inside} \quad S_{R'} \\
(1.2)\qquad &\boldsymbol{m} = -\frac{1}{2}\boldsymbol{r} + \mu\,, \ \mu(r) = \mu_0(r)\cdot\Phi\left(\frac{r-R'}{R-R'}\right) \quad \text{for} \quad R' \leq r \leq R \\
&\boldsymbol{m} = -\frac{1}{2}\boldsymbol{r} \qquad \text{outside} \quad S_R\,.
\end{aligned}
$$

Hence our problem reduces to the simple problem for $r > R$. In the domain $D_R = \Omega \setminus \Omega_R$ the field $\boldsymbol{m}$ defines a direction field that is close to the radial direction, due to the smallness of μ_0. We can find flux lines for such a field, and use them as integration paths to construct T, as we shall see in section 2.

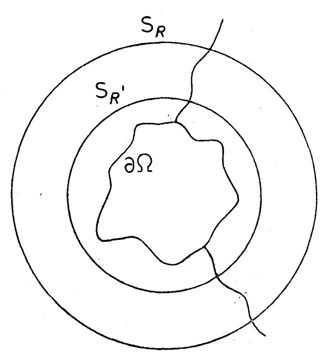

Fig. 1.

To this aim, for example, r itself can be used as a parameter: in this way one can see that the flux lines are solutions of an ordinary system of differential equations, regular (i.e. in C^1) everywhere in D_R.

In D_R we can use as coordinates for $\boldsymbol{x}$ the spherical angular coordinates σ of the crossing point of the flux line through $\boldsymbol{x}$ with S_R, and the arc length s

on the flux line from x_R to x (fig. 2). As the direction of the flux lines is close to the radial one, the derivatives of s with respect to r are greater and not very different from 1. From the preceding remarks it is not difficult to see that the Jacobian of the transformation between the coordinates (s, σ) and the ordinary polar coordinates (r, σ') is bounded and non-vanishing (the difficulty that arises at poles can be overcome by dividing into subdomains and using suitably rotated coordinates). Once $\boldsymbol{m}$ is defined by (1.2), we devise a procedure to construct

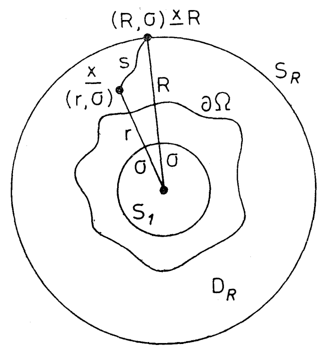

Fig. 2.

a function $\tilde{u}$ in Ω that satisfies the equation $-\boldsymbol{m} \cdot \nabla T + T = \tilde{u}$, with the same boundary condition as in (1.1).

Such a procedure assumes as a starting point that the function T is known. Then, starting from u, we reconstruct T.

The solution of our problem is uniquely determined if the composition of the two procedures has just one fixed point, which occurs if it is a contraction in a suitably defined metric space.

2. Extension of u over the whole Ω

Let us define $\tilde{H}^1_R(\Omega)$ as the space of the functions $f \in H^1_{loc}(\Omega)$, harmonic outside a sphere centred at the origin of radius R and vanishing at infinity. We see immediately that $\nabla f \in L^2(\Omega)$; we can choose as a norm

$$\|f\|_{\tilde{H}^1_R(\Omega)} = \int_\Omega |\nabla f|^2 \, dx . \tag{2.1}$$

Such a norm can be proved to be equivalent to

$$|||f||| = \|f\|_{H^1(D)} = \|\nabla f\|_{L^2(\Omega \setminus D)}$$

where D is any bounded subdomain of Ω, obtained by taking the intersection of Ω with a star-shaped domain whose boundary encloses $\partial\Omega$.

Now let $T \in \tilde{H}^1_R(\Omega)$ be harmonic on the whole Ω; moreover we require the coefficients of Y_{1m}/r^2 to vanish. Our aim is to investigate the properties of the function $\tilde{u} = -\boldsymbol{m} \cdot \nabla T + T$ in Ω, where $\boldsymbol{m}$ is defined as in sec. 1.

Taking into account the harmonic character of T we find

$$\begin{aligned} \Delta\tilde{u} &= \Delta(-\boldsymbol{m} \cdot \nabla T + T) = \Delta\left(\frac{1}{2}\, r\, \frac{\partial T}{\partial r} + T - \mu \cdot \nabla T\right) = \\ &= \Delta(-\mu \cdot \nabla T) = -\sum_i (\Delta\mu_i)\, \partial_i T - 2 \sum_{i,j} \partial_j \mu_i\, \partial_i\, \partial_j T = \\ &= \sum_i (\Delta\mu_i)\, \partial_i T - 2 \sum_{i,j} \partial_j (\partial_j \mu_i\, \partial_i T) \equiv AT\,. \end{aligned} \tag{2.2}$$

We have used the commutation rule $\Delta\left(\frac{1}{2}\, r \frac{\partial}{\partial r} + I\right) = \left(\frac{1}{2}\, r \frac{\partial}{\partial r} + 2\,I\right)\Delta$ (cfr. Sansò [8]). Obviously $\Delta\tilde{u} = 0$ in Ω_R; AT has support in D_R and belongs to $H^{-1}(D_R)$.

Hence $\tilde{u}$ is a solution of the Dirichlet problem

$$\begin{cases} \Delta\tilde{u} = AT & \text{in } \Omega \qquad AT \in H^{-1}(D_R) \\ u\,|_{\partial\Omega} = u & \qquad\qquad\quad u \in H^{1/2}(\partial\Omega). \end{cases} \tag{2.3}$$

We write $\tilde{u} = v + h$, where

$$a)\begin{cases} \Delta v = AT & \text{in } \Omega \\ v\,|_{\partial\Omega} = 0 \end{cases} \qquad b)\begin{cases} \Delta h = 0 & \text{in } \Omega \\ h\,|_{\partial\Omega} = u. \end{cases} \tag{2.4}$$

Now, it is well known that $b)$ has a unique solution belonging to $\tilde{H}^1_R(\Omega)$ when $u \in H^{1/2}(\partial\Omega)$ and $\Omega \in \mathcal{N}^{(0),1}$. The same is true for problem $a)$, as one can easily see, by applying the Kelvin transformation.

Let $v \in \tilde{H}^1_R(\Omega)$ be the solution of $a)$. As its asymptotic behaviour at infinity is at least of order $1/r$, then we can write:

$$\begin{aligned} \|v\|^2_{\tilde{H}^1_R(\Omega)} &= \int_\Omega |\nabla v|^2\, dx = -\int_\Omega v\Delta v\, dx = \\ &= -\int_{D_R} v \sum_i \Delta\mu_i\, \partial_i\, T dx - 2 \int_{D_R} \sum_{ij} \partial_j\, v\, \partial_j\, \mu_i\, \partial_i\, T dx \end{aligned} \tag{2.6}$$

(as μ vanishes outside D_R).

Now,

$$(2.7)\quad \left| \int_{D_R} \sum_{ij} \partial_j v \, \partial_j \, \mu_i \, \partial_i \, T dx \right| \leq \bar{\mu} \sum_i \left(\int_{D_R} (\partial_i T)^2 dx \right)^{1/2} \sum_j \left(\int_{D_R} (\partial_j v)^2 dx \right)^{1/2} \leq$$

$$\leq 3 \, \bar{\mu} \, \| T \|_{\tilde{H}^1_R(\Omega)} \, \| v \|_{\tilde{H}^1_R(\Omega)}$$

where $\bar{\mu} = \sup | \partial_j \mu_i |$.

On the other hand,

$$(2.8)\quad \left| \int_{D_R} v \sum_i \Delta\mu_i \, \partial_i \, T dx \right| \leq \sum_i \left(\int_{D_R} (\partial_i T)^2 dx \right)^{1/2} \cdot \left(\int_{D_R} v^2 \, \Delta \, \mu_i^2 \, dx \right)^{1/2} .$$

As $v \in H^1(D_R)$, from Sobolev's immersion theorem $v \in L^6(D_R)$, and

$$\| v \|_{L^6(D_R)} \leq c \, \| v \|_{H^1(D_R)} \leq c' \, \| v \|_{\tilde{H}^1_R(\Omega)} \text{ ; hence}$$

$$(2.9)\quad \left| \int_{D_R} v^2 \, \Delta\mu_i^2 \, dx \right| \leq \left(\int_{D_R} v^6 dx \right)^{1/3} \left(\int_{D_R} \Delta\mu_i^3 \, dx \right)^{3/2} \leq c'^2 \, \hat{\mu}^2 \, \| v \|^2_{\tilde{H}^1_R(\Omega)}$$

where $\hat{\mu}^2 = \sup_i \left(\int_{D_R} \Delta\mu_i^3 \, dx \right)^{3/2}$.

Introducing (2.9) into (2.8) we obtain

$$(2.10)\quad \left| \int_{D_R} v \sum_i \Delta\mu_i \, \partial_i \, T dx \right| \leq \sqrt{3} \, c' \, \hat{\mu} \, \| v \|_{\tilde{H}^1_R(\Omega)} \, \| T \|_{\tilde{H}^1_R(\Omega)} .$$

Summing up, using (2.7) and (2.10) we get the result

$$(2.11)\quad \| v \|_{\tilde{H}^1_R(\Omega)} \leq \bar{c}\mu_o \, \| T \|_{\tilde{H}^1_R(\Omega)}$$

where $\mu_o = \max(\bar{\mu}, \hat{\mu})$ and $\bar{c}$ is independent of T.

Hence we can write $v = GAT$, where the operator GA is linear and bounded from $\tilde{H}^1_R(\Omega)$ to $\tilde{H}^1_R(\Omega)$.

Note that the harmonic character of T is used only to obtain (2.2); once the operator A is defined and is proved to vanish outside S_R, the solution of (2.3) and the definition of the operator G do not require that T is harmonic.

3. Bounds for T in terms of $\tilde{u}$

Now, our aim is to reconstruct T from $\tilde{u}$ inverting the relation $-\boldsymbol{m}\cdot\nabla T + T = \tilde{u}$, and to see which are the properties of such T when $\tilde{u}$ fulfils the same properties as the function $\tilde{u}$ defined in the previous section. First we consider the case that $\tilde{u}$ is a C^1 function in $\tilde{H}^1_R(\Omega)$, whose restriction to Ω_R belongs to the space $HH'^1(\Omega_R)$ of harmonic functions vanishing at infinity, with zero first degree harmonic components. Let $\Omega_{\bar{R}}$ and $D_{\bar{R}}$ be defined as Ω_R and D_R in section 1, with $\bar{R} > R$. T is defined in $\Omega_{\bar{R}}$ as in Sansò [8] from the equation $T + \frac{1}{2} r \frac{\partial T}{\partial r} = \tilde{u}$. As $\tilde{u}$ is harmonic down to S_R, the same is true for T, which therefore is a regular function on $S_{\bar{R}}$; moreover T lacks first degree components in the spherical harmonic expansion outside S_R, and depends continuously in $HH'^1(\Omega_R)$ on $\tilde{u}$ in $HH'^1(\Omega)$. In $D_{\bar{R}}$, T is defined as

$$(3.1)\quad T(\boldsymbol{x}) = \left[T(\boldsymbol{x}_{\bar{R}}(\boldsymbol{x})) - \int_0^{s(\boldsymbol{x})} d\tau \frac{\tilde{u}(\tau, \boldsymbol{x}_{\bar{R}}(\boldsymbol{x}))}{m(\tau, \boldsymbol{x}_{\bar{R}}(\boldsymbol{x}))} \exp\left(-\int_0^{\tau} \frac{dt}{m(t, \boldsymbol{x}_{\bar{R}}(\boldsymbol{x}))}\right)\right] \cdot \exp \int_0^{s(\boldsymbol{x})} \frac{d\tau}{m(\tau, \boldsymbol{x}_{\bar{R}}(\boldsymbol{x}))} \qquad (m = |\boldsymbol{m}|).$$

The integration path is the flux line through $\boldsymbol{x}$ (see section 1); $\boldsymbol{x}_{\bar{R}}(\boldsymbol{x})$ is the crossing point with $S_{\bar{R}}$. $s(\boldsymbol{x})$ is the arc length of the flux line from $\boldsymbol{x}_R$ to $\boldsymbol{x}$. The flux line is parametrized by the arc length τ. It is easily seen that (3.1) is a solution of the ordinary differential equation

$$(3.2)\qquad -m\frac{\partial T}{\partial s} + T \equiv -\boldsymbol{m}\cdot\nabla T + T = \tilde{u}\,.$$

We remark that all the properties of the field $\boldsymbol{m}$ obtained in section 1 in D_R are easily extended to $D_{\bar{R}}$; in $D_{\bar{R}} \setminus D_R$ we have $\boldsymbol{m} = -\frac{1}{2}\boldsymbol{r}$ and (3.1) reduces to equation (10) of Sansò [8]. Our aim is to prove that $T \in H^1(D_{\bar{R}})$ and that $\| T \|_{H^1(D_{\bar{R}})} \leq k \| u \|_{\tilde{H}^1_R(\Omega)}$.

The factor

$$(3.3a)\qquad g(\boldsymbol{x}) = \exp \int_0^{s(\boldsymbol{x})} \frac{d\tau}{m(\tau, \boldsymbol{x}_{\bar{R}}(\boldsymbol{x}))}$$

as well as the coefficient of u in the integrand,

$$(3.3b)\qquad f(\tau, \boldsymbol{x}) = \frac{1}{m(\tau, \boldsymbol{x}_{\bar{R}}(\boldsymbol{x}))} \exp\left(-\int_0^{\tau} \frac{dt}{m(t, \boldsymbol{x}_{\bar{R}}(\boldsymbol{x}))}\right)$$

is a differentiable function of x in $D_{\bar{R}}$; therefore it is bounded together with its derivatives with respect to x_j in $D_{\bar{R}}$. We have already pointed out that T is a regular function on $S_{\bar{R}}$; we know from Sansò [8] that

$$\|T\|_{H^1(S_{\bar{R}})} \le c_1 \|\tilde{u}\|_{\tilde{H}^1_R(\Omega)} . \tag{3.4}$$

Hence we can write

$$\int_{D_{\bar{R}}} d^3 x \, | T(x_{\bar{R}}(x)) |^2 = \int_{S_1} d\sigma \int_0^{\bar{s}(\sigma)} ds \, | \det J \, | \, | T(x_{\bar{R}}) |^2 \le \tag{3.5}$$

$$\le J_{\max} s_{\max} \int_{S_1} d\sigma \, | T(x_{\bar{R}}) |^2$$

where S_1 is the unit sphere, $\bar{s}(\sigma)$ is the arc length of the flux line from the point on $S_{\bar{R}}$ with angular coordinates σ to $\partial\Omega$, and $s_{\max}$ its maximum value for $\sigma \in S_1$, J is the Jacobian matrix from the cartesian coordinates to (s, σ) and $J_{\max}$ the maximum value of its determinant over D_R. An analogous bound can be obtained for $\nabla T(x_{\bar{R}}(x))$. In fact $\nabla T(x_{\bar{R}}(x)) = \nabla_{\bar{R}} T \dfrac{\partial (x_{\bar{R}})}{\partial (x)}$, where $\nabla_{\bar{R}} T$ denotes the vector of derivatives with respect to the coordinates on $S_{\bar{R}}$, which is in $L^2(S_{\bar{R}})$ by (3.4), and $\dfrac{\partial (x_{\bar{R}})}{\partial (x)}$ is the Jacobian matrix of these coordinates, that is bounded. Consequently, it is proved that $T(x_{\bar{R}}(x)) \in H^1(D_R)$; moreover its norm in $H^1(D_{\bar{R}})$ is bounded by $c_2 \|u\|_{\tilde{H}^1_R(\Omega)}$.

Now, let us examine the integral in (3.1).

$$I(x) = \int_0^s d\tau \, \frac{\tilde{u}}{m} \exp\left(- \int_0^\tau \frac{dt}{m} \right) . \tag{3.6}$$

It is not difficult to prove that $I(x)$ is bounded in terms of $\|\tilde{u}\|_{L^2(D_{\bar{R}})}$. Moreover we have

$$\frac{\partial I}{\partial x_j} = \frac{u(x)}{m(x)} \exp\left(- \int_0^{s(x)} \frac{dt}{m(t, x_{\bar{R}}(x))} \right) \frac{\partial s}{\partial x_j} + \tag{3.7}$$

$$+ \int_0^{s(x)} d\tau \left\{ \tilde{u}(\tau, x_{\bar{R}}(x)) \frac{\partial f}{\partial x_j}(\tau, x_{\bar{R}}(x)) + \right.$$

$$\left. + \frac{\partial \tilde{u}}{\partial x_j}(\tau, x_{\bar{R}}(x)) f(\tau, x_{\bar{R}}(x)) \right\} \equiv I_1(x) + I_2(x) .$$

It is also easily proved that I_1 can be bounded by $c_3 \| \tilde{u} \|_{L^2(D_{\overline{R}})}$; the same is true for the first term in the integral I_2. As for the second term in I_2, it is possible to obtain a bound in terms of $\| \nabla u \|_{L^2(D_{\overline{R}})}$. Summing up, we can conclude that the integral in the definition (3.1) of $T(x)$ belongs to $H^1(D_{\overline{R}})$, so that $T|_{D_{\overline{R}}} \in H^1(D_{\overline{R}})$.

Now we wish to release the C^1 assumption for $\tilde{u}$ in $D_{\overline{R}}$ and require only $\tilde{u}|_{D_{\overline{R}}} \in H^1(D_{\overline{R}})$, taking into account that $\tilde{u}$ is harmonic down to S_R, which ensures the connection between the two sides of $S_{\overline{R}}$. In this case one can simply define

$$T(x) = [T(x_{\overline{R}}(x)) + (L\tilde{u})(x)] \exp \int_0^{s(x)} \frac{d\tau}{m(\tau, x_{\overline{R}}(x))} \tag{3.8}$$

where $L : H^1(D_{\overline{R}}) \to H^1(D_{\overline{R}})$ is the unique linear continuous operator that extends the integral (3.6) from the dense subspace $C^1(D_{\overline{R}})$ onto $H^1(D_{\overline{R}})$. T still satisfies equation (3.2), in the distribution sense. In conclusion we can state that T is in $\tilde{H}^1_R(\Omega)$, and write $T = J\tilde{u}$, where J is a linear continuous operator from $\tilde{H}^1_R(\Omega)$ into itself. We recall once more that, by definition, the function $T = J\tilde{u}$ outside $S_{\overline{R}}$ is harmonic and lacking first degree harmonic components.

4. Regularization

Our aim is now to prove that $\nabla T|_{\partial\Omega} \in L^2(\partial\Omega)$. Let first T be defined by (3.1), that can be written

$$T(x) = [T(x_{\overline{R}}(x)) - I(x)]\, g(x) \tag{4.1}$$

where $I(x)$ is defined by (3.6), $g(x)$ by (3.3a). We have remarked in sec. 3 that $g(x)$ is bounded together with its derivatives with respect to x_j in $D_{\overline{R}}$.

We have

$$\frac{\partial T}{\partial x_j} = [T(x_{\overline{R}}(x) - I(x)] \frac{\partial g}{\partial x_j} + \left[\frac{\partial}{\partial x_j} T(x_{\overline{R}}(x)) - \frac{\partial I}{\partial x_j}\right] g(x) \tag{4.2}$$

and we have to study this function on the surface $\partial\Omega$.

We recall that $\partial\Omega$ is the boundary of a star-shaped domain of class $\mathcal{N}^{(0),1}$. Since every point of $\partial\Omega$ is reached by a flux line starting at $S_{\overline{R}}$, it is easy to see that $\partial\Omega$ must satisfy an equation of the form $r = r(\sigma)$, where σ is defined as in sec. 1 and $r(\sigma)$ is a Lipschitz function. Consequently, if $\Phi(x)$ is any non-negative integrable function on $\partial\Omega$, we have

$$\int_{\partial\Omega} \Phi(x)\, d\Sigma \leq k \int_{S_1} \Phi(r(\sigma))\, d\sigma . \tag{4.3}$$

Then, by arguments similar to the ones yielding (3.5) we can write

$$\int_{\partial\Omega} | \mathrm{T}(x_{\bar{\mathrm{R}}}(x)) |^2 \, \mathrm{d}\Sigma \leq k \int_{\mathrm{S}_1} \mathrm{d}\sigma \, | \mathrm{T}(x_{\bar{\mathrm{R}}}) |^2 \leq k' \, \| \tilde{u} \|_{\tilde{\mathrm{H}}^1_{\mathrm{R}}(\Omega)}$$

and a similar inequality for $\frac{\partial}{\partial x_j} \mathrm{T}(x_{\bar{\mathrm{R}}}(x))$; thus we conclude that both $\mathrm{T}(x_{\bar{\mathrm{R}}}(x))$ and $\frac{\partial}{\partial x_j} \mathrm{T}(x_{\bar{\mathrm{R}}}(x))$ belong to $\mathrm{L}^2(\partial\Omega)$, with a norm that can be bounded in terms of $\| \tilde{u} \|_{\tilde{\mathrm{H}}^1_{\mathrm{R}}(\Omega)}$.

As for $\mathrm{I}(x)$ we can write

$$\int_{\partial\Omega} | \mathrm{I}(x) |^2 \, \mathrm{d}\Sigma \leq k \int_{\mathrm{S}_1} \mathrm{d}\sigma \left| \int_0^{\bar{s}(\sigma)} \mathrm{d}\tau \, \frac{\tilde{u}}{m} \exp\left(- \int_0^{\tau} \frac{\mathrm{d}t}{m} \right) \right|^2 \leq \tag{4.4}$$

$$\leq k \int_{\mathrm{S}_1} \mathrm{d}\sigma \left(\int_0^{\bar{s}(\sigma)} | f(\tau, x) |^2 \, \mathrm{d}\tau \right) \left(\int_0^{\bar{s}(\sigma)} | \tilde{u} |^2 \, \mathrm{d}\tau \right) \leq$$

$$\leq k_1 \int_{\mathrm{S}_1} \mathrm{d}\sigma \int_0^{\bar{s}(\sigma)} | \tilde{u} |^2 \, \mathrm{d}\tau \leq k_2 \, \| \tilde{u} \|^2_{\mathrm{L}^2(\mathrm{D}_{\mathrm{R}})} .$$

To obtain a bound for $(\partial \mathrm{I} / \partial x_j)$ on $\partial\Omega$ we use equation (3.7). By the same procedure as in (4.4), we easily obtain for the integral in (3.7)

$$\int_{\partial\Omega} \mathrm{d}\Sigma \, | \mathrm{I}_2(x) |^2 \leq k_3 \, \| \tilde{u} \|^2_{\mathrm{H}^1(\mathrm{D}_{\mathrm{R}})} . \tag{4.5}$$

As for the first term, we take into account that the coefficient of $\tilde{u}(x)$ is bounded in D_{R} and on $\partial\Omega$, and that $u(x) \in \mathrm{H}^{1/2}(\partial\Omega)$; then we can write

$$\int_{\partial\Omega} \mathrm{d}\Sigma \, | \mathrm{I}_1(x) |^2 \leq k_4 \, \| u \|^2_{\mathrm{L}^2(\partial\Omega)} \leq k_4 \, | u |^2_{\mathrm{H}^{1/2}(\partial\Omega)} \leq k_5 \, \| \tilde{u} \|^2_{\mathrm{H}^1_{\mathrm{R}}(\Omega)} . \tag{4.6}$$

Recalling (4.2), from (4.4), (4.5), (4.6) we obtain that $\nabla \mathrm{T}$ has a trace on $\partial\Omega$ that belongs to $\mathrm{L}^2(\partial\Omega)$, with

$$\| \nabla \mathrm{T} \|^2_{\mathrm{L}^2(\partial\Omega)} \leq k_0 \, \| \tilde{u} \|^2_{\tilde{\mathrm{H}}^1_{\mathrm{R}}(\Omega)} . \tag{4.7}$$

If, more generally, T is defined by (3.8), the same result can be obtained by taking the limit of a sequence of regular approximating functions $\tilde{u}_n$.

5. Solution of the problem

Let us define $V = \{w \in \tilde{H}^1_R(\Omega) \mid , w|_{\partial\Omega} = u\}$; V is a closed manifold in $\tilde{H}^1_R(\Omega)$. Let us take $\hat{u} \in V$; $\hat{u}|_{\Omega_R} = \Sigma u_{nm} \left(\frac{R}{r}\right)^{n+1} Y_{nm}$. Let us define $\hat{u}_1 = P\hat{u} = \hat{u} - \sum_j u_{1j} \left(\frac{R}{r}\right)^2 Y_{1j}$. Now let us take $T = J\hat{u}_1 = JP\hat{u}$, where J is the operator introduced at the end of sec. 3. As we have seen there, $T \in \tilde{H}^1_R(\Omega)$, $T|_{\Omega_R} \in HH'^1(\Omega_R)$, and the equation $-m \cdot \nabla T + T = \hat{u}_1$ holds. Therefore

$$\Delta\hat{u} = \Delta\hat{u}_1 = \frac{1}{2} r \frac{\partial}{\partial r}(\Delta T) + 2\Delta T + AT \qquad \text{in} \quad \Omega \tag{5.1}$$

as T is not necessarily harmonic in D_R.

On the other hand, from such a function T we can construct $\tilde{u} = v + h = GAT + h$, satisfying problem (2.4), so that $\tilde{u}$ belongs to V too.

Summing up, we can write

$$\tilde{u} = h + GAT = h + GAJP\hat{u}. \tag{5.2}$$

The linear operator GAJP is continuous from $\tilde{H}^1_R(\Omega)$ into itself; moreover, the norm of GA is proportional to the constant μ_0 introduced in (2.11). Now we can see from the construction of $\mu(r)$ in sec. 1 that, if we start from a sufficiently small field μ_0 in (1.2) and R is chosen large enough, the constant $\bar{\mu}$ and $\hat{\mu}$, and hence μ_0, turn out to be suitably small, for instance in such a way that

$$\|GA\| \quad \|JP\| < 1. \tag{5.3}$$

Consequently the transformation $\hat{u} \to \tilde{u}$ defined in (5.2) is a contraction from V into itself and there it has a unique fixed point. Let $\tilde{u}$ be such a point; since $\Delta\tilde{u} = AT$, and $\tilde{u} = \hat{u}$, it follows from (5.1) that

$$\frac{1}{2} r \frac{\partial}{\partial r}(\Delta T) + 2\Delta T = 0 \qquad \text{in} \quad \Omega. \tag{5.4}$$

From $T \in \tilde{H}^1_R(\Omega)$ we already know that $\Delta T = 0$ in Ω_R; hence the unique solution of (5.4) is $\Delta T = 0$ all over Ω.

Thus we have found a function T that is harmonic in Ω and satisfies the boundary condition

$$-m_0 . \nabla T + T|_{\partial\Omega} = \hat{u}_1|_{\partial\Omega} = \hat{u}|_{\partial\Omega} - \Sigma u_{1j} A_j \tag{5.5}$$

(A_j defined as in (1.1)). Then the b.v.p. (1.1) is satisfied, with the choice $a_j = -u_{1j}$. At last we prove that the solution is unique. In fact, if (T_1, a_1)

and (T_2, a_2) both satisfy (1.1), and we define $\hat{T} = T_2 - T_1$, $\hat{a} = a_2 - a_1$, we have

$$(5.6) \qquad \begin{cases} \Delta\hat{T} = 0 \quad \text{in} \quad \Omega \\ -m \cdot \nabla\hat{T} + \hat{T}\,|_{\partial\Omega} = \hat{a} \cdot \mathbf{A} \\ \hat{T} = \dfrac{\hat{\alpha}}{r} + 0\,(r^{-3})\,. \end{cases}$$

Then $h = \sum_j \hat{a}_{1j}\, Y_{1j} \left(\dfrac{R}{r}\right)^2$. Applying the operator JP to the equation $\tilde{u} = h + \mathrm{GAJP}\tilde{u}$, and observing that in our case $\mathrm{JP}h = 0$, we find

$$(5.7) \qquad \hat{T} = \mathrm{JPGA}\hat{T}.$$

By the same arguments used above, we can state that the norm of GA can be chosen so small that $\| \mathrm{JP} \| \cdot \| \mathrm{GA} \| < 1$. Hence the only possible solution of (5.7) is $\hat{T} = 0$.

6. CONCLUSIONS

The effort to settle the analysis of the linear problem of Molodenskii under reasonable regularity conditions on the boundary data has been successful, showing a sufficient regularity of the solution, as required in the Introduction (Note I). In particular, the relevant oblique derivative problem has been studied for a boundary satisfying a cone condition, which required a considerable application of non standard techniques, all related to the analysis of the functional properties of quasi-radial integral operators. The net result is that the unknown function T can be constructed according to sec. 5 as $T = \mathrm{JP}\hat{u}$, where $\hat{u}$ satisfies the suitable fixed point equation (5.2). Since $\hat{u}$ belongs to $\tilde{H}^1_R(\Omega)$, T is so regular as to admit a square integrable gradient on the boundary $\partial\Omega$, as it is shown in (4.7): this was devised as the minimal admissible regularity condition for the solution of the linear problem (1.1), because the displacement vector relating the actual physical surface of the earth to the approximated reference surface $\partial\Omega$ depends linearly on ∇T, in the linearized formulation. It seems worth mentioning that the present result, though interesting from the analytical point of view, still shows the extreme difficulty of Molodenskii's problem in that half derivative is lost in the present analysis, with respect to the more classical results for the problems of Dirichlet and Neumann.

This has also the effect that we cannot attack the solution of the non-linear problem by a simple iterative method. This peculiar fact seems to be characteristic in Molodenskii's problem and has forced Hörmander to apply a rather heavy regularization technique to achieve his result [1] for the nonlinear problem.

However, even in the field of the pure linear problem of Molodenskii there are still two important open questions:

— is the numerical value of the constant μ_0 (cfr. (2.11)), for a realistic earth model, so small as to guarantee that the existence condition (5.3) is satisfied?

— is the linear problem of Molodenskii solvable for any given regular isozenithal field (pointing always towards the earth) and for any given boundary term u?

These questions deserve future investigations, before the linear problem of Molodenskii can be claimed to be solved.

References

[1] Hörmander L. (1976) – *The boundary problems of physical geodesy.* « Arch. Rat. Mech. An. », *62*, 1-52.

[2] Krarup T. – *Letters on Molodentky's problem.* Unpublished manuscript.

[3] Lions J.L. et Magenes E. (1968) – *Problèmes aux limites non homogènes and applications.* Dunod, Paris.

[4] Nečas J. (1967) – *Les methodes directes dans la théorie des équations elliptiques.* Masson et C., Paris and Academia, Prague.

[5] Sacerdote F. and Sansò F. (1983) – *The current situation in the linear problem of Molodenskii.* Note I. « Rend. Acc. Naz. dei Lincei, Cl. di Sc. Fis. Mat. e Nat », sez. VIII, *75*, 119-126.

[6] Sansò F. (1977) – *The Geodetic Boundary Value Problem in Gravity Space.* « Atti Acc. Naz. Lincei, Mem. Cl. Sc. Fis. Mat. Nat. », ser. VIII, *14*, 41-97.

[7] Sansò F. (1981) – *Recent Advances in the Theory of the Geodetic Boundary Value Problem.* « Rev. Geoph. Space Phys. », *19*, 437-449.

[8] Sansò F. (1981) – *The point on the simple Molodensky's problem.* « Rend. Acc. Naz. dei Lincei, Cl. di Sc. Fis. Mat. e Nat. », ser. VIII, *71*, 87-94.

[9] Witsch K.J. (1978) – *The Geodetic Boundary Value Problem : A Uniqueness Result.* Rep. Inst. für Angew. Math. der Univ. Bonn.