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**On the abstract Cauchy problem in the case of
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Analisi matematica. — *On the abstract Cauchy problem in the case of constant domains.* Nota di PAOLO ACQUISTAPACE e BRUNELLO TERRENI, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si studiano esistenza, unicità e regolarità delle soluzioni strette, classiche e forti $u \in C([0, T], E)$ dell'equazione di evoluzione non autonoma $u'(t) - A(t)u(t) = f(t)$ con il dato iniziale $u(0) = x$, in uno spazio di Banach E . Gli operatori $A(t)$ sono generatori infinitesimali di semi-gruppi analitici ed hanno dominio indipendente da t e non necessariamente denso in E . Si danno condizioni necessarie e sufficienti per l'esistenza e la regolarità hölderiana della soluzione e della sua derivata.

0. INTRODUCTION

Let E be a Banach space, and $\{A(t)\}_{t \in [0, T]}$ a family of closed linear operators on E ; suppose that for each $t \in [0, T]$ $A(t)$ generates an analytic semi-group and has a domain $D(A(t)) = D(A(0))$ which does not depend on t and is possibly not dense in E . We consider the following Cauchy problem:

$$(P) \quad \begin{cases} u'(t) - A(t)u(t) = f(t), & t \in [0, T] \\ u(0) = x \\ x \in E, f: [0, T] \rightarrow E \text{ prescribed.} \end{cases}$$

Many authors have studied this problem (see the references). The case of variable domains has been treated in [1]; thus, in this paper we will assume the same hypotheses of Tanabe [12]; more precisely, we suppose the following:

I) For each $t \in [0, T]$ $A(t)$ is a closed linear operator on the Banach space E , with domain $D(A(t)) = D(A(0))$ independent on t , generating the analytic semi-group $\{e^{\xi A(t)}\}_{\xi \geq 0}$: in particular

(i) there exists $\theta^0 \in]\pi/2, \pi]$ such that

$$\rho(A(t)) \supseteq \Sigma_{\theta^0} := \{z \in \mathbb{C} : z = \rho e^{i\theta}, \rho \in [0, \infty[, \theta \in]-\theta^0, \theta^0[\}, \forall t \in [0, T];$$

(ii) there exists $M > 0$ such that $\forall t \in [0, T]$

$$\|R(\lambda, A(t))\|_{\mathcal{L}(E)} \leq \frac{M}{|\lambda|} \quad \forall \lambda \in \Sigma_{\theta^0} - \{0\}, \quad \|A(t)^{-1}\|_{\mathcal{L}(E)} \leq M$$

(*) Nella seduta del 14 gennaio 1984.

II) There exist $\alpha \in]0, 1[$ and $K > 0$ such that

$$\|1 - A(t)A(\tau)^{-1}\|_{\mathcal{L}(E)} \leq K |t - \tau|^\alpha \quad \forall t, \tau \in [0, T].$$

Let us specify now what we mean as a solution of Problem (P). Denote by $C([0, T], E)$ (resp. $C(]0, T], E)$) the space of continuous functions $[0, T] \rightarrow E$ (resp. $]0, T] \rightarrow E$), and set

$$D^0 = \{u \in C([0, T], E) : u(t) \in D(A(0)) \quad \forall t \in [0, T] \quad \text{and} \\ u', A(\cdot)u(\cdot) \in C([0, T], E)\},$$

$$D = \{u \in C([0, T], E) : u(t) \in D(A(0)) \quad \forall t \in]0, T] \quad \text{and} \\ u', A(\cdot)u(\cdot) \in C(]0, T], E)\}.$$

Now we define our solutions.

DEFINITION 0.1. $u : [0, T] \rightarrow E$ is a strict solution of Problem (P) if $u \in D$ and

$$u'(t) - A(t)u(t) = f(t) \quad \forall t \in [0, T], \quad u(0) = x.$$

DEFINITION 0.2. $u : [0, T] \rightarrow E$ is a classical solution of Problem (P) if $u \in D^0$ and

$$u'(t) - A(t)u(t) = f(t) \quad \forall t \in]0, T], \quad u(0) = x.$$

DEFINITION 0.3. $u : [0, T] \rightarrow E$ is a strong solution if $u \in C([0, T], E)$ and there exists $\{u_n\}_{n \in \mathbb{N}} \subseteq D$ such that

- (i) $u_n \rightarrow u$ in $C([0, T], E)$,
- (ii) $u'_n - A(\cdot)u_n(\cdot) =: f_n \rightarrow f$ in $C([0, T], E)$,
- (iii) $u_n(0) =: x_n \rightarrow x$ in E .

Remark 0.4. By definition every strict solution is also a classical and a strong one whereas it is not obvious that a classical solution is also a strong one; this is true under hypotheses I and II. The situation in the case of variable domains is different (see Remark 6.5 of [1]).

In this paper we give existence, uniqueness and regularity results for strict, classical and strong solutions of (P); in addition we give a representation formula for the solutions, without passing through the construction of the fundamental solution; our method is mainly inspired by the techniques of Da Prato-Grisvard [5].

Our formula can be heuristically derived by the following argument: we look for a solution of this kind:

$$(0.1) \quad u(t) := e^{tA(0)} x + \int_0^t e^{(t-s)A(s)} g(s) ds, \quad t \in [0, T],$$

where g is a suitable (integrable) function. Of course when $A(t) \equiv A$ this formula with $g = f$ gives the ordinary mild solution of (P); since $A(t)$ depends on t now, it is natural to expect that f has to be suitably modified. Taking the formal derivative of (0.1) we get

$$\begin{aligned} u'(t) &= A(t) u(t) + (A(0) - A(t)) e^{tA(0)} x + g(t) + \\ &+ \int_0^t (A(s) - A(t)) e^{(t-s)A(s)} g(s) ds, \end{aligned}$$

hence if we want (0.1) to be a solution of (P) we must choose g such that

$$(0.2) \quad g(t) + \int_0^t K(t, s) g(s) ds = f(t) - K(t, 0) x, \quad t \in [0, T],$$

$$K(t, s) := (A(s) - A(t)) e^{(t-s)A(s)}.$$

Denote by K the integral operator

$$(0.3) \quad K\varphi(t) := \int_0^t K(t, s) \varphi(s) ds;$$

then the representation formula for the solution of Problem (P) is formally given by

$$(F) \quad u(t) = e^{tA(0)} x + \int_0^t e^{(t-s)A(s)} (1 + K)^{-1} (f - K(\cdot, 0) x)(s) ds, \quad t \in [0, T].$$

We also study the maximal time regularity of the strict solution: in other words we prove that the solution u has Hölder continuous derivative in $[0, T]$ if f is Hölder continuous with the same exponent, provided the vectors x and $f(0)$ satisfy a suitable compatibility condition which is necessary and sufficient. Our condition generalizes the results of Poulsen [8], Sinestrari [9], [10], and Da Prato-Sinestrari [7].

Finally we give some space regularity results for the strong solution, i.e. we prove that the solution is continuous with values in the interpolation spaces

between $D(A(0))$ and E , provided x belongs to the same space. Regarding classical solutions, we obtain an «a priori» estimate for $A(t)u(t)$ which is of interest in the study of the quasi-linear version of (P). More results, with all proofs, will appear in a forthcoming paper.

1. PRELIMINARIES

If $\beta \in [0, 1[$, denote by $C_\beta([0, T], E)$ (resp. $C_\beta([0, T], E)$) the space of functions $u \in C([0, T], E)$ such that $t \rightarrow t^\beta u(t) \in L^\infty(0, T, E)$ (resp. $C([0, T], E)$) (note that $C_0([0, T], E) = C([0, T], E)$). $C_\beta([0, T], E)$ is obviously a Banach space with norm

$$\|u\|_{C_\beta([0, T], E)} = \|t^\beta u(\cdot)\|_{L^\infty(0, T, E)},$$

having $C_\beta([0, T], E)$ as a closed subspace.

For each $\sigma \in]0, 1[$ we define the Hölder space $C^\sigma([0, T], E) = \{f \in C([0, T], E) : \|f(t) - f(\tau)\|_E = o(t - \tau)^\sigma \text{ as } t - \tau \rightarrow 0^+\}$ and its closed subspace $h^\sigma([0, T], E) = \{f \in C^\sigma([0, T], E) : \|f(t) - f(\tau)\|_E = o(t - \tau)^\sigma \text{ as } t - \tau \rightarrow 0^+\}$; we shall also use the spaces

$$C^\sigma([0, T], E) := \bigcap_{\varepsilon \in]0, T[} C^\sigma([\varepsilon, T], E),$$

$$h^\sigma([0, T], E) := \bigcap_{\varepsilon \in]0, T[} h^\sigma([\varepsilon, T], E).$$

The interpolation spaces $D_{A(0)}(\theta, \infty)$ and $D_{A(0)}(\theta)$ between $D(A(0))$ and E ($\theta \in]0, 1[$) can be defined in several equivalent ways (see Butzer-Berens [3], Da Prato-Grisvard [6]). In particular we have

$$\begin{aligned} D_{A(0)}(\theta, \infty) &= \{x \in E : \sup_{t>0} t^{-\theta} \|e^{tA(0)}x - x\|_E < \infty\} = \\ &= \{x \in E : \sup_{t>0} t^{1-\theta} \|A(0)e^{tA(0)}x\|_E < \infty\} = \\ &= \{x \in E : \sup_{\lambda \in \rho(A(0))} |\lambda|^\theta \|A(0)R(\lambda, A(0))x\|_E < \infty\}, \end{aligned}$$

and similarly

$$\begin{aligned} D_{A(0)}(\theta) &= \{x \in E : \lim_{t \rightarrow 0^+} t^{-\theta} \|e^{tA(0)}x - x\|_E = 0\} = \\ &= \{x \in E : \lim_{t \rightarrow 0^+} t^{1-\theta} \|A(0)e^{tA(0)}x\|_E = 0\} = \\ &= \{x \in E : \lim_{\substack{\lambda \in \rho(A(0)) \\ |\lambda| \rightarrow \infty}} |\lambda|^\theta \|A(0)R(\lambda, A(0))x\|_E = 0\} \end{aligned}$$

$D_{A(0)}(\theta, \infty)$ is a Banach space with norm $\|x\|_\theta := \sup_{t > 0} t^{-\theta} \|e^{tA(0)} x\|_E$; moreover the quantities $\sup_{t > 0} t^{1-\theta} \|A(0) e^{tA(0)} x\|_E$ and $\sup_{\lambda \in \rho(A(0))} |\lambda|^\theta \|A(0) R(\lambda, A(0)) x\|_E$ define two norms on $D_{A(0)}(\theta, \infty)$ which are equivalent to $\|x\|_\theta$.

Clearly $D(A(0)) \subseteq D_{A(0)}(\theta, \infty) \subseteq E$ with continuous inclusions (when $D(A(0))$ is equipped with the graph norm); in addition $D_{A(0)}(\theta, \infty)$ —which is a closed subspace of $D_{A(0)}(\theta, \infty)$ —coincides with the closure of $D(A(0))$ in $D_{A(0)}(\theta, \infty)$ (see [3]).

The analytic semi-groups $\{e^{\xi A(t)}\}_{\xi \geq 0, t \in [0, T]}$, have a standard representation by a Dunford integral: if $\theta \in]\pi/2, \pi]$, set $\gamma := \gamma^0 \cup \gamma^- \cup \gamma^+$, where

$$\gamma^0 := \{\lambda \in \mathbf{C} : |\lambda| = 1, |\arg \lambda| \leq \theta\},$$

$$\gamma^\pm := \{\lambda \in \mathbf{C} : \lambda = \rho e^{\pm i\theta}, \rho \geq 1\};$$

then under hypothesis I we have

$$A(t)^n e^{\xi A(t)} x = \frac{1}{2\pi i} \int_{\gamma} \lambda^n e^{\xi \lambda} R(\lambda, A(t)) d\lambda, \forall \xi > 0, \forall t \in [0, T], \forall n \in \mathbf{N},$$

the integrals being absolutely convergent.

In addition, under hypotheses I, II it is clear that

$$\|K(t, s)\|_{\mathcal{L}(E)} = \|(1 - A(t)A(s)^{-1})A(s)e^{(t-s)A(s)}\|_{\mathcal{L}(E)} \leq \frac{C}{(t-s)^{1-\alpha}},$$

$$0 \leq s < t \leq T.$$

Hence it is easy to show that the operator K defined in (0.3) is continuous on $L^p(0, T, E)$ and $C_\beta([0, T], E)$, $p \in [1, \infty]$, $\beta \in [0, 1[$, and $(1 + K)^{-1}$ exists and is also continuous in such spaces. As a direct consequence, formula (F) defines an element u of $C_0([0, T], E)$; moreover $u \in C([0, T], E)$ if and only if $x \in D(A(0))$.

2. MAIN RESULTS

Here is a list of our main results, whose proofs will appear in a forthcoming paper.

THEOREM 2.1. (Necessary conditions). *Under hypotheses I, II let $x \in E$ and $f \in C([0, T], E)$.*

- (i) *If u is a strict solution of (P), then $x \in D(A(0))$ and $A(0)x + f(0) \in \overline{D(A(0))}$.*
- (ii) *If u is a classical (resp. a strong) solution of (P), then $x \in \overline{D(A(0))}$.*

THEOREM 2.2. (Uniqueness and continuous dependence on data). *Under hypotheses I, II let $x \in E$ and $f \in C([0, T], E)$. Then Problem (P) has at most one strict (resp. classical or strong) solution. Moreover if u is such a solution, then*

$$\|u(t)\|_E \leq C \left\{ \|x\|_E + \int_0^t \|f(s)\|_E ds \right\} \quad \forall t \in [0, T].$$

THEOREM 2.3. (Existence of the strict solution). *Under hypotheses I, II let $x \in D(A(0))$ and $f \in C^\sigma([0, T], E)$, $\sigma \in]0, \alpha]$ (resp. $h^\sigma([0, T], E)$, $\sigma \in]0, \alpha[$), and suppose that $A(0)x + f(0) \in \overline{D(A(0))}$. Then the function u defined by (F) is the strict solution of (P) and belongs to $C^{1,\sigma}([0, T], E)$ (resp. $h^{1,\sigma}([0, T], E)$).*

THEOREM 2.4. (Maximal time regularity of the strict solution). *Under hypotheses I, II let $x \in D(A(0))$ and $f \in C^\sigma([0, T], E)$, $\sigma \in]0, \alpha]$ (resp. $h^\sigma([0, T], E)$, $\sigma \in]0, \alpha[$), and suppose that u is a strict solution of (P). Then $u \in C^{1,\sigma}([0, T], E)$ (resp. $h^{1,\sigma}([0, T], E)$) if and only if $A(0)x + f(0) \in D_{A(0)}(\sigma, \infty)$ (resp. $D_{A(0)}(\sigma)$).*

THEOREM 2.5. (Existence of the classical solution). *Under hypotheses I, II let $x \in \overline{D(A(0))}$ and $f \in C([0, T], E) \cap C^\sigma([0, T], E)$, $\sigma \in]0, 1[$ (resp. $C([0, T], E) \cap h^\sigma([0, T], E)$). Then the function u defined by (F) is the classical solution of (P) and belongs to $C^{1,\sigma}([0, T], E)$ (resp. $h^{1,\sigma}([0, T], E)$). In addition if $x \in D_{A(0)}(\beta, \infty)$ and $f \in C^\sigma([0, T], E)$, $\beta, \sigma \in]0, 1[$, then*

$$\|A(t)u(t)\|_E \leq C \left\{ \frac{1}{t^{1-\beta}} \|x\|_\beta + \|f\|_{C^\sigma([0, T], E)} \right\} \quad \forall t \in]0, T].$$

THEOREM 2.6. (Existence of the strong solution). *Under hypotheses I, II let $x \in \overline{D(A(0))}$ and $f \in C([0, T], E)$. Then the function u defined by (F) is the strong solution of (P) and belongs to $C^\sigma([0, T], E) \forall \sigma \in]0, 1[$. In addition $u \in C^\sigma([0, T], E)$ (resp. $h^\sigma([0, T], E)$), $\sigma \in]0, 1[$, if and only if $x \in D_{A(0)}(\sigma, \infty)$ (resp. $D_{A(0)}(\sigma)$).*

THEOREM 2.7. (Space regularity of the strong solution). *Under hypotheses I, II let $x \in \overline{D(A(0))}$, $f \in C([0, T], E)$ and let u be a strong solution of (P). Then $u \in C^{1-\theta}([0, T], D_{A(0)}(\theta)) \forall \theta \in]0, 1[$, and moreover*

- (i) *$u \in C_0([0, T], D_{A(0)}(\theta))$ (resp. $C([0, T], D_{A(0)}(\theta))$), $\theta \in]0, 1[$, if and only if $x \in D_{A(0)}(\theta, \infty)$ (resp. $D_{A(0)}(\theta)$);*
- (ii) *if $x \in D_{A(0)}(\theta, \infty)$ (resp. $D_{A(0)}(\theta)$), $\theta \in]0, 1[$, then $u \in C^{0-\beta}([0, T], D_{A(0)}(\beta))$ (resp. $h^{0-\beta}([0, T], D_{A(0)}(\beta))$) $\forall \beta \in]0, \theta[$.*

Remark 2.7 Existence and uniqueness of the strict solution is still guaranteed if, instead of Hölder continuity, we suppose that $f \in C([0, T], E)$ and its oscillation $\omega(\cdot)$ satisfies

$$(2.1) \quad \int_0^T \frac{\omega(s)}{s} ds < \infty.$$

This generalizes a result of Crandall-Pazy [4]. On the other hand the only assumption $f \in C([0, T], E)$ is not sufficient for existence, even if $x = 0$ and $A(t) \equiv A$ (see Baillon [2], Travis [14], Da Prato-Grisvard [7]).

Similarly, we have existence and uniqueness of the classical solution if $f \in C_\theta([0, T], E)$ for some $\theta \in [0, 1[$ and in addition the oscillations $w_\varepsilon(\cdot)$ of $f|_{[t_\varepsilon, T]}$ satisfy (2.1).

Finally we note that the preceding results apply to many concrete situations; two examples will be described in our forthcoming paper.

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