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A uniqueness theorem for viscous flows on exterior domains with summability assumptions on the gradient of pressure.

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Fisica matematica. — *A Uniqueness Theorem for Viscous Flows on Exterior Domains with Summability Assumptions on the Gradient of Pressure* (*). Nota di GIOVANNI P. GALDI e PAOLO MAREMONTI presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — In questa Nota si fornisce un teorema di unicità per soluzioni regolari delle equazioni di Navier-Stokes in domini esterni. Tale teorema non richiede che le velocità tendano ad un prefissato limite all'infinito, mentre il gradiente di pressione è supposto essere di q-ma potenza sommabile nel cilindro spazio-temporale ($q \in (1, \infty)$). Questo risultato non può essere ulteriormente generalizzato al caso $q = \infty$, a causa di noti controesempi.

1. In a well known paper of 1960 [13] D. Graffi proved the uniqueness of a smooth solution $(\boldsymbol{v}, p)$ (1) to the Navier-Stokes equations in an exterior domain Ω, in the class of solutions $(\boldsymbol{v} + \boldsymbol{u}, p + \pi)$ such that $\boldsymbol{u}, \boldsymbol{v}$ and their first spatial derivatives are uniformly bounded in $\Omega \times [0, T]$ $(T > 0)$ and $|\pi(x, t)| \leq K/|x|$ $(K = \text{const} > 0)$ for $|x|$ sufficiently large. This theorem is remarkable in that, unlike the uniqueness results known at that time, it did not require that the velocity field tend to a prescribed limit at large spatial distances. Subsequently, several authors reconsidered this kind of problem and, by using also different techniques, improved Graffi's theorem by weakenning the assumptions on $\boldsymbol{u}, \boldsymbol{v}$ and π [4–10, 12–17]. In particular, in [5] it has lately been shown that the assumption on the pressure can be relaxed to π uniformly bounded in $\Omega \times [0, T]$, i.e., without requiring any infinitesimality at infinity. However, more recently, it has been noted in [11] that it could be more meaningful from a physical point of view to give uniqueness theorems by making hypotheses on $\nabla\pi$ *directly* instead of π. Precisely, in [11] it is proved that when Ω is the whole space (i.e., $\Omega = R^3$) and in a class of velocities larger than that considered by Graffi (2), uniqueness holds provided

$$\text{(I)} \qquad \int_0^T \int_\Omega |\nabla\pi|^q \, d\Omega \, dt < \infty \qquad \text{for some} \quad q \in (1, \infty).$$

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(1) Here $\boldsymbol{v} = \boldsymbol{v}(x, t)$ and $p = p(x, t)$ denote the velocity and pressure fields, respectively.

(2) Actually, it is assumed that $\boldsymbol{u}$ and $\boldsymbol{v}$ have *only* first spatial derivatives uniformly bounded in $\Omega \times [0, T]$.

Moreover, in [11] it is also shown by means of counterexamples that (I) is the « best possible » in the sense that it can *not* be improved to $q = \infty$. Unfortunately, this result has an undesired feature, namely, it is proved only for the Cauchy problem ($\Omega = R^3$). Further, it seems that the methods employed in [11] can not be easily generalized to the more interesting case of a non-empty (compact) boundary.

In this note we prove that the uniqueness theorem given in [11] continues to hold also when Ω is a generic (sufficiently smooth) exterior domain, in the class of velocities such that $\boldsymbol{v}$, $\nabla \boldsymbol{v}$ and $\nabla \boldsymbol{u}$ are uniformly bounded. Precisely, we prove the following theorem.

THEOREM. *Let* Ω *be* C^2*-smooth and let* $(\boldsymbol{v}, p)$, $(\boldsymbol{v} + \boldsymbol{u}, p + \pi)$ *be two classical solutions to the Navier-Stokes equations in* $\Omega \times [0, T]$ $(T > 0)$ *subject to the same body force and the same boundary data. Then, if for some positive constant* C (3)

$$| \boldsymbol{v} | , | \nabla \boldsymbol{u} | , | \nabla \boldsymbol{u} | \leq C$$

and, moreover, (I) *holds, then* $\boldsymbol{u}$ *vanishes identically in* $\Omega \times [0, T]$.

Obviously, also in our case condition (I) can not be weakened to $q = \infty$, because of the counterexample given [11] (cf. also [15], and our Remark 2).

It is remarkable that the above theorem is proved quite easily just by suitably coupling well known L^p- estimates holding for solutions to the heat equation [14] and to the Navier-Stokes equations in exterior domains [16], [11].

2. As is well known, the « difference motion » $(\boldsymbol{u}, \pi)$ satisfies the following initial-boundary value problem (for the sake of simplicity, we set the kinematical viscosity coefficient equal to one)

$$\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \underset{\sim}{\boldsymbol{v}} \cdot \nabla \underset{\sim}{\boldsymbol{u}} + \underset{\sim}{\boldsymbol{u}} \cdot \nabla \underset{\sim}{\boldsymbol{u}} \equiv - \nabla \pi + \Delta \boldsymbol{u} \quad \text{in} \quad \Omega_T = \Omega \times [0, T) \\ (1) \qquad &\nabla \cdot \boldsymbol{u} = 0 \\ &\boldsymbol{u}(x, 0) = 0 \qquad x \in \Omega \\ &\boldsymbol{u}(y, t) = 0 \qquad (y, t) \in \partial\Omega \times [0, T]. \end{aligned}$$

The proof of the uniqueness theorem will be achieved through several steps. To this end, we need to introduce the space $W_q(\Omega_t)$ $(\Omega_t \equiv \Omega \times [0, t]$

(3) Throughout this note, by C we denote a generic positive constant whose numerical value is unessential to our aims and it may have several values in a single computation. For example we may have, in the same line, $2\,C \leq C$.

$t \leq T$, cf. [16]) of functions $w : \Omega_t \to R$ such that w, $\partial w/\partial t$, $\partial w/\partial x_i$, $\partial^2 w/\partial x_i \partial x_j$ $(i, j = 1, 2, 3)$ belong to the Lebesgue space $L^q(\Omega_t)$. We have

LEMMA 1. *Consider the problem* $(t \leq T)$

$$\begin{aligned} &\frac{\partial w}{\partial t} + v \cdot \nabla w - \Delta w = F \qquad in \quad \Omega_t \\ (2) \qquad &w(x, 0) = 0 \quad x \in \Omega \\ &w(y, t) = 0 \quad (y, t) \in \partial\Omega \times [0, t]. \end{aligned}$$

Then, given $F \in L^q(\Omega_t)$ $(q > 1)$ *there exists at least one solution* $w \in W_q(\Omega_t)$ *to problem* (2). *Moreover, this solution is unique among (sufficiently smooth) solutions* $w' \in L^q(\Omega_t)$.

Proof. The existence result is classical and well known [14]. The uniqueness proof is not explicitly given in [14]. However, it is very simple and will be sketched here. Denote by $\phi \in C^2(\Omega)$ a « cut-off » function such that

$$\begin{aligned} (3) \qquad &\phi(x) = 1 \quad \text{if} \quad |x| \leqq R, \ \phi(x) = 0 \quad \text{if} \ |x| \geqq 2R \\ &|\nabla\phi| \leqq C \quad , \quad |\Delta\phi| \leqq C. \end{aligned}$$

Multiply, then, equation (2) with $F = 0$ by $\phi w |w|^{q-2}$ and integrate by parts over Ω. One shows easily

$$(4) \qquad \int_\Omega \phi |w|^q d\Omega \leqq \int_0^T \int_\Omega |\underset{\sim}{v} \cdot \nabla\phi + \Delta\phi| |w|^q d\Omega ds, \qquad \tau \leqq t.$$

Therefore, since v is uniformly bounded in Ω_T, uniqueness follows taking into account $(3)_2$, letting $R \to \infty$ and then employing Gronwall's lemma.

LEMMA 2. *Assume that with respect to a fixed spherical coordinate system* (r, γ) *with the origin at a given point of* R^3 *it holds*

$$\begin{aligned} (5) \qquad &|v_r|, |u_r| \leqq Cr, |u| \leqq Cr^m \qquad (m \geq 0, r \geq \bar{r}) \\ &|\nabla v| \leqq C \ |\nabla u| \leqq Cr^k \qquad (k \geq 0, r \geq \bar{r}) \end{aligned}$$

where v_r *and* u_r *are the radial components of* v *and* u, *respectively. Then, for all* $q > 1$

$$\nabla\pi \in [L^q(\Omega_T)]^3 \qquad implies \quad u \in [L^q(\Omega)]^3$$

and the following estimate holds

$$\int_{\Omega} | \boldsymbol{u}(x, t)|^q \, d\Omega \leqq C \int_0^t \int_{\Omega} | \nabla\pi |^q \, d\Omega \, ds \qquad t \leq T . \tag{6}$$

Proof. See [11], Lemma 1.

LEMMA 3. *Consider the linearized Navier-Stokes problem*

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v} = - \nabla\pi + \Delta \boldsymbol{w} + \boldsymbol{f}$$

$$\begin{array}{ll} \nabla \cdot \boldsymbol{w} = 0 & \text{in } \Omega_t \\ \boldsymbol{w}(x, 0) = 0 & x \in \Omega \\ \boldsymbol{w}(y, t) = 0 & (y, t) \in \partial\Omega \times [0, t) . \end{array}$$

Then, given $\boldsymbol{f} \in [L^q(\Omega_t)]^3$ $(q > 1)$, *There exists one and only one solution* $\boldsymbol{w} \in [W^q(\Omega_t)]^3$. *Moreover this solution satisfies the estimate*

$$\int_0^t \int_{\Omega} | \nabla\pi |^q \, d\Omega \, ds \leqq C \int_0^t \int_{\Omega} (|\boldsymbol{f}|^q + |\boldsymbol{w}|^q) \, d\Omega \, ds.$$

Proof. See [16], Theorem 4.2.

We are in a position to prove the uniqueness theorem. We set $\boldsymbol{F} = - \nabla\pi - \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{v}$. By assumption and by Lemma 2 it follows that $\boldsymbol{F} \in [L^q(\Omega_t)]^3$ $t \leq T$ (4). Hence, by Lemma 1 we deduce $\boldsymbol{u} \in [W_i(\Omega_t)]^3$, $t \leq T$. Putting $\boldsymbol{f} = - \boldsymbol{u} \,.\, \nabla \boldsymbol{u}$, in virtue of Lemmas 2 and 3 we thus obtain the estimate

$$\int_0^t \int_{\Omega} | \nabla \pi |^q \, d\Omega \, ds \leqq C \int_0^t \int_{\Omega} | \boldsymbol{u} |^q \, d\Omega \, ds. \tag{7}$$

Employing (7) in (6), there follows

$$\int_{\Omega} | \boldsymbol{u}(x, t) |^q \, d\Omega \, ds \leqq C \int_0^t \int_{\Omega} | \boldsymbol{u}(x, s) |^q \, d\Omega \, ds, \; t \leq T$$

which, by Gronwall's lemma, gives uniqueness.

(4) Notice that $| \nabla \underset{\sim}{u} | \leq C$ implies $| u | \leq C$.

Remark 1. The theorem continues to hold for certain (smooth) domains with non-compact boundary (e.g., a half space). This type of domain must satisfy simultaneously the requirements of [14], p. 294, and those of [1]. Actually, Theorem 1 of [1] generalized to suitable domains with non-compact boundary the results stated in Lemma 3.

Remark 2. The hypothesis $q < \infty$ in the theorem cannot be weakened to $q = \infty$, in general, as the following counterexample shows (with $\Omega = \mathrm{R}^3$)

$$\boldsymbol{v} = 0 \qquad p = 0$$

$$\boldsymbol{u} = (t, 0, 0) \qquad \pi = -x_1 .$$

References

[1] Bogovskii M.E. (1980) – *On the L^p-theory of the Navier-Stokes system for unbounded domains with non compact boudaries*, « Soviet Math. Doklady », *22*, 809-814.

[2] Cannon J.R. and Knightly G.H. (1970) – *Some continuous dependence theorems for viscous fluid motions*, « SIAM J. Appl. Math. », *18*, 627-640.

[3] Edmunds E. (1963) – *On the uniqueness of viscous flows*, « Arch. Rational Mech. Anal. », *14*, 171-176.

[4] Fabrizio M. (1979) – *Problemi di unicità per le equazioni di Navier-Stokes in domini non limitati*, « Arch. Rational Mech. Anal. », *68*, 171-178.

[5] Galdi G.P. and Maremonti P. – *A Uniqueness Theorem for Viscous Fluid Motions in Exterior Domains*, « Arch. Rational Mech. Anal. », to appear.

[6] Galdi G.P. and Rionero S. (1976) – *A uniqueness theorem for hydrodinamic flows in unbounded domains*, « Ann. Mat. Pura App. », *108*, 361-366 (1976).

[7] Galdi G.P. and Rionero S. (1976) – *On the uniqueness of viscous fluid motions*, « Arch. Rational Mech. Anal. », *62*, 295-301.

[8] Galdi G.P. and Rionero S. (1979) – *The weight function approach to uniqueness of*

[9] Galdi G.P. and Rionero S. (1979) – *Continuous dependence theorems for Navier-Stokes equations in unbounded domains by the weight function method*, « Q. Jl. Mech. Appl. Math. », *32*, 149-161.

[10] Galdi G.P. and Rionero S. (1979) – *A priori estimates, continuous dependence and stability for solutions to Navier-Stokes equations on exterior domains*, « Riv. Mat. Univ. Parma », *5*, 533-566.

[11] Galdi G.P. and Rionero S. (1983) – *On the best counditions on the gradient of pressure for uniqueness of viscous flows in the whole space*, « Pacific J. Math. », *104*, 77-83.

[12] Galdi G.P. and Rionero S. (1983) – *Local estimates and stability of viscous flows in exterior domains*, « Arch. Rational Mech. Anal. », *81*, 333-347.

[13] Graffi D. (1960) – *Sul teorema di unicità nella dinamica dei fluidi*, « Ann. Mat. Pura App. », *50*, 379-388.

[14] Ladyzhenskaya O.A., Solonnikov V.A. and Ural'Tseva N.N. (1968) – *Linear and quasi-linear equations of parabolic type*, « American Math. Soc. Providence ».

[15] Muratori P. (1971) – *Teoremi di unicità per un problema relativo alle equazioni di Navier-Stokes*, « Boll. U.M.I. », *4*, 592-613.

[16] Solonnikov V.A. (1977) – *Estimates for solutions of non stationary Navier-Stokes*, « J. Soviet Math. », *8*, 467-529.

[17] Straughan B. (1977) – *Uniqueness and continuous dependence theorems for the conduction-diffusion solution to the Boussinesq equations on an exterior domain*, « J. Math. Anal. App. », *57*, 203-234.