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Fisica matematica. — *On some viscoelastic models* (*). Nota di PASQUALE RENNO, presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — Sia $\mathscr{B}_n$ un sistema linearmente viscoelastico, omogeneo ed isotropo, caratterizzato dalla funzione di memoria $g_n(t) = \sum_1^n {}_k B_k \exp(-\beta_k t)$, tipica di numerosi polimeri solidi. Si dimostra che la soluzione fondamentale E_n dell'operatore integrodifferenziale che descrive i moti di $\mathscr{B}_n$ è, in ogni punto del suo supporto, maggiorata da quella relativa ad un opportuno solido standard $\mathscr{B}_1$. Di conseguenza, è possibile applicare all'analisi qualitativa dei moti di $\mathscr{B}_n$ alcuni risultati stabiliti in [10], quali proprietà asintotiche, principi di massimo e teoremi di approssimazione per problemi di perturbazione singolare.

0. The well known *creep* representation of the one-dimensional linear motions of an isotropic, intrinsically homogeneous viscoelastic system $\mathscr{B}$ is

$$\text{(0.1)}\qquad L\,u = c^2\,u_{xx} - u_{tt} - \int_0^t g(t-\tau)\,u_{\tau\tau}(x,\tau)\,d\tau = -\tilde{f}\ ,$$

where $u(x,t)$ is the single non-vanishing component of the displacement field from an undeformed homogeneous reference configuration $\mathscr{R}$ and $\tilde{f}$ represents a source term depending on the body forces and on the past history of stress [11]. Further $g(t)$ denotes the ratio $\dot{J}(t)/J(0)$, where $J(t)$ is the creep compliance, while $c^2 = [\rho J(0)]^{-1}$, where ρ is the constant mass density in $\mathscr{R}$.

In a previous paper [11] the fundamental solution $<E, \chi>$ $(\chi \in \mathscr{S}(R^2))$ of the operator L for an arbitrary $g(t)$ has been constructed and consequently the initial value problem related to (0.1) has been explicitly solved.

Moreover, in the meaningful case that $g > 0$, $\dot{g} < 0$ on R^+, appropriate estimates of E allow to compare—on every bounded initial time interval $[0, T]$—the behaviour of $\mathscr{B}$ with that well known of the media $\mathscr{B}_0$ and $\mathscr{B}_T$ characterized by the *constant memories* $g(0)$ and $g(T)$. But, when t is large, these estimates

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fail and further properties of g are necessary, as Dafermos [3] has proved even when $x \in [0, 1]$.

To continue this analysis and to have an idea of the asymptotic phenomena, we now deal with the following case:

$$g = g_n(t) = \sum_{1}^{n} {}_k \, B_k \, e^{-\beta_k t} , \tag{0.2}$$

where n is quite arbitrary and the *retardation frequencies* β_k are of course strictly positive and, without loss of the generality, such that $\beta_1 < \beta_2 < \ldots < \beta_n$. As for the constants B_k, we will assume that $B_k > 0$ $(k = 1, \ldots, n)$ in order that $g_n(t) > 0$ and $\dot{g}_n(t) < 0$ (1). In this way $g_n(t)$ verifies also the convexity assumption considered by Dafermos.

According to well-known Muntz and Schwartz's theorems [1] concerning the uniform approximation of $C(R^+)$ functions by Dirichlet polynomials, the case of (0.2) is little restrictive. Furthermore (0.2) describes wide classes of practically important viscoelastic models as n, B_k and β_k are determined to fit experimental curves for $g(t)$ to any desired degree of approximation. So, as an example, many actual polymeric materials with their broad molecular weight distribution and their highly complex internal structure can be truly represented [2] by means of (0.2).

According to the results of [11], the fundamental solution $<E, \chi>$ of L is associated with a $C^\infty(\mathring{\Gamma})$ function E whose support is the forward characteristic cone Γ (sect. 2). To specify the dependence on n, β_k, B_k we denote $\mathscr{B}$ with $\mathscr{B}_n$, J with J_n and

$$E = E_n(\beta_1, \ldots\ldots \beta_n ; B_1, \ldots\ldots, B_n) \qquad n \geq 1 . \tag{0.3}$$

Further, let

$$a_n = \sum_{1}^{n} {}_k \, B_k/\beta_k \qquad , \qquad \chi_n = \prod_{2}^{n} {}_k \, (\beta_k/\beta_1)^2 . \tag{0.4}$$

The case $n = 1$ corresponds to the *linear standard solid* $\mathscr{B}_1$ whose behaviour has been rigorously evaluated in [10] for all $t \geq 0$ and $x \in R^k$ $(k = 1, 2, 3)$. The related fundamental solution is induced by a function $E_1(\beta_1 ; B_1)$ which is bounded also when $t \to \infty$ and its derivatives are rapidly decreasing functions. Therefore, when $n > 1$, it should be very useful to estimate E_n in terms of E_1.

This aim is achieved by means of the following Theorem.

(1) Obviously this is the most usual and simple sufficient condition in order that $g > 0$ and $\dot{g} < 0$ on R^+. Other conditions could be deduced by Čebyšev inequality [4].

THEOREM 0.1 *When the response function* $g(t)$ *is given by* (0.2) (*with all* β_k, B_k *positive*), *then the fundamental solution* $\langle E_n, \chi \rangle$ *of the operator* L *is induced by a never negative* $C^\infty(\mathring{\Gamma})$ *function* E_n *which satisfies the estimate*

$$(0.5) \qquad 0 < E_n(\beta_1 \ldots \beta_n ; B_1 \ldots B_n) < \chi_n E_1(\beta_1 ; \beta_1 a_n),$$

everywhere in Γ *and whatever* n *may be.*

Consequently, the rigorous analysis of asymptotic properties and singular perturbation problems for $\mathscr{B}_n$—whatever n may be—can be achieved by means of the basic properties of the appropriate standard linear model $\mathscr{B}_1$ defined by

$$(0.6) \qquad g_1(t) = B_1 e^{-\beta_1 t} \qquad \text{with} \quad B_1 = \beta_1 a_n .$$

This comparison model is physically meaningful for the following reasons. As β_1 is the smallest retardation frequency, then $\mathscr{B}_1$ is related just to the *obliviator* $\exp(-\beta_1 t)$ and to the *characterstic time* $\tau_1 = \beta_1^{-1}$ of $\mathscr{B}_n$ defined just by the longest retardation time. Moreover $\mathscr{B}_n$ and $\mathscr{B}_1$ satisfy the same hypotheses of fading memory. In fact, by (0.2)-(0.4)-(0.6) one deduces obviously

$$(0.7) \qquad \int_0^\infty g_n(t)\, dt = \int_0^\infty g_1(t)\, dt = a_n$$

and the value of this integral, as it is well known [6, 8], is decisive for the asymptotic analysis of hereditary equations and the compatibility with the principle of fading memory.

At last we observe that (0.1)-(0.2) can be reduced to a partial differential equation of order $n+2$, typical of a *wave hierarchy* with $n+1$ characteristic speeds [13]. Then, as we will see in Sect 4, Theorem 0.1 implies that the most meaningful speeds which affects *eventually* the wave behavior of $\mathscr{B}_n$ are those typical of $\mathscr{B}_1$, i.e.:

a) The faster speed $c = [\rho J_n(0)]^{-1/2}$ of the wave front, which is related to small precursor waves.

b) The speed $c_0 = [\rho J_n(\infty)]^{-1/2} < c$ connected with the regime value $J_n(\infty)$ of the creep compliance and related to the main signal which prevails at large t.

So, one can prove that very slow viscoelastic processes ($\tau_1 \to 0$) are quasi-elastic with modulus $J_n(\infty)$ and that when t is large ($t > \tau_1$) the most conspicious phenomenon in $\mathscr{B}_n$ is a diffusion process connected with the lowest speed c_0. Rigorous estimates of such phenomera can be obtained by means of Theorem 0.1 and Theorem 6.3 of [10].

1. In [11] we have considered the one-dimensional motions of an isotropic homogeneous body $\mathscr{B}$ with a linearly viscoelastic behaviour of creep type given by

$$e(t) = J(0)\left[\sigma(t) + \int_{-\infty}^{t} g(t-\tau)\,\sigma(\tau)\,d\tau\right], \tag{1.1}$$

where $\sigma(x, t)$, $e(x, t)$ are the single non-vanishing components of the stress and strain tensors, while

$$g(t) = \dot{J}(t)/J(0), \tag{1.2}$$

being $J(t)$ the creep compliance. Now, in the case we deal with, by (0.2)-(1.2) one has $J(t) = J_n(t)$ with

$$J_n(t) = J_n(0)\left[1 + \sum_{k=1}^{n} \frac{B_k}{\beta_k}(1 - e^{-\beta_k t})\right], \tag{1.3}$$

where β_k^{-1} are the *retardation times* and $J_n(0)\,B_k/\beta_k$ represent the *elastic compliances*. As β_k and B_k are strictly positive constants, by (1.3) one has

$$J_n(\infty) = J_n(0)\left[1 + \sum_{k=1}^{n} \frac{B_k}{\beta_k}\right] > J_n(0). \tag{1.4}$$

Further, the characteristic speed of the wave front is given by $c^2 = [\rho\, J_n(0)]^{-1}$.

2. According to [11], we denote with Γ the forward characteristic cone $\{(t, x) : t > 0, |x| \le ct\}$, with $\eta(t)$ the Heaviside function and with $\mathscr{L}$ the Laplace operator. Further let s be the parameter of the $\mathscr{L}_t$-transformation and let

$$\hat{h}(s) = \mathscr{L}[-\dot{g}(t)], \quad r = |x|/c, \quad g(0) = g_0. \tag{2.1}$$

Then the fundamental solution $E(x, t)$ of the operator L is defined [11] by

$$E(x, t) = (2c)^{-1}\,\eta(t-r)\,(A' + A'') \tag{2.2}$$

with

$$A' = e^{-\frac{1}{2}g_0 t}\, I_0\left(\tfrac{1}{2} g_0 \sqrt{t^2 - r^2}\right) \tag{2.3}$$

$$A'' = \pi^{-1}\int_0^{\pi} d\theta \int_r^{t} e^{-g_0 z}\, H(z, t-u)\, du, \tag{2.4}$$

where $z = \frac{1}{2}(u - \cos\theta \sqrt{u^2 - r^2})$, I_0 is the modified Bessel function of the first kind and

(2.5) $$H(z, t) = \mathscr{L}^{-1}[e^{z\hat{h}(s)} - 1] .$$

In [11], $g(t)$ being quite arbitrary, $H(z, t)$ has been constructed by means of a series of iterated convolutions. But when $g = g_n(t)$ is defined by (0.2) the sum of this series can be explicitly computed as follows.

By (2.1)-(0.2) one has

$$\hat{h}(s) = \sum_1^n {}_k \frac{B_k \beta_k}{s + \beta_k} = \hat{h}_n(s)$$

and therefore by (2.5)

$$\mathscr{L} H(z, t) = \hat{H}_n(z, s) = \exp\left[z \sum_{k=1}^n \frac{B_k \beta_k}{s + \beta_k}\right] - 1 .$$

Now, if one puts

$$\hat{\varphi}_k = \exp\left(z \frac{B_k \beta_k}{s + \beta_k}\right) - 1 \qquad k = 1, \ldots, n$$

it results ([9] p. 244)

(2.6) $$\varphi_k(z, t) = \mathscr{L}^{-1} \hat{\varphi}_k = e^{-\beta_k t} (B_k \beta_k z / t)^{1/2} \quad I_1(2\sqrt{B_k \beta_k z t}) .$$

On the other hand, setting

$$\hat{H}_r = \exp\left(z \sum_{k=1}^r \frac{B_k \beta_k}{s + \beta_k}\right) - 1 ,$$

the recurrence formula holds

$$\hat{H}_1 = \hat{\varphi}_1 , \quad \hat{H}_r = \hat{\varphi}_r + (1 + \hat{\varphi}_r) \hat{H}_{r-1} \quad r = 2, \ldots, n .$$

Consequently $H = H_n$ is determined by reccurrence as follows

(2.7) $$H_1 = \varphi_1 , \quad H_r = \varphi_r + H_{r-1} + \varphi_r * H_{r-1} \quad (r = 2, \ldots, n)$$

where the functions φ_r are defined in (2.6). By (2.7) it is easy to deduce that

(2.8) $$H = H_n = \sum_1^n {}_{k_1} \varphi_{k_1} + \sum {}_{k_1 k_2} \varphi_{k_1} * \varphi_{k_2} + \cdots ,$$

where the sums are computed according to the simple combinations of the indices $k_1, k_2, \ldots, k_n$.

In conclusion, observing that I_1 is an analytic function, we can state that:

« *When the response function* $g(t)$ *is given gy* (0.2), *the fundamental solution* $E(x, t)$ *of the operator* L *is the strictly positive value* $C^\infty(\Gamma)$ *function defined by* (2.2)-(2.3)-(2.4)-(2.8) ».

3. On the analogy of (0.3), when it is necessary, we will specify the dependence of A″, H, ... on the parameters B_k, β_k by setting $A'' = A''_n(\beta_1 \ldots]$ β_n; $B_1 \ldots B_n)$ etc. ...

Now, when $n = 1$, one has the case of the standard linear solid where, if $g_1(t) = Be^{-\beta t}$, it results

$$E_1(\beta ; B) = (2c)^{-1} \eta (t - r) (A'_1 + A''_1) \tag{3.1}$$

with

$$A'_1(B) = e^{-\frac{1}{2}Bt} I_0 (\tfrac{1}{2} B \sqrt{t^2 - r^2}) \tag{3.2}$$

$$A''_1(\beta ; B) = \pi^{-1} \int_0^\pi d\theta \int_r^t e^{-Bz-\beta(t-u)} \sqrt{\frac{B\beta z}{t-u}}\, I_1 (z \sqrt{B\beta z (t-u)})\, du . \tag{3.3}$$

In [10] it has been proved that E_1 is bounded also when $t \to \infty$ and its derivatives are rapidly decreasing functions. By means of these and other properties one achieves to solve rigorously for $\mathscr{B}_n$ various questions such as asymptotic behaviour, singular perturbation problems, diffusion of waves and maximum principles.

In order to generalize these results to the case of n arbitrary, we now give the

Proof of Theorem 0.1. Let $d_i = B_i/\beta_i$ be and let

$$\tilde{\varphi}_i = \beta_i^{-2} e^{-B_i z} \varphi_i = e^{-\beta_i (t+d_i z)} \sqrt{\frac{d_i z}{t}}\, \beta^{-1} I_1 (2 \beta_i \sqrt{d_i z t}) . \tag{3.4}$$

These functions, as functions of β_i, are not increasing as

$$\frac{\partial}{\partial \beta_i} \tilde{\varphi}_i = e^{-\beta_i (t+d_i z)} \quad \beta_i^{-1} (d_i z/t)^{1/2} \cdot$$

$$\cdot [2 \sqrt{d_i zt}\, I_2 (2 \beta_i \sqrt{d_i zt}) - (t + d_i z) I_1 (2 \beta_i \sqrt{d_i zt})] \leq 0$$

being $I_2(u) \leq I_1(u)$ for all $u \geq 0$. Consequently, as $\beta_1 < \beta_i \, (i \geq 2)$ one has

$$e^{-B_i z} \, \beta_i^{-2} \, \varphi_i < \beta_1^{-2} \, e^{-\beta_1 (t + d_i z)} \, \lambda(d_i, t) \qquad (i \geq 2) \tag{3.5}$$

where the function

$$\lambda(d_i, t) = \beta_1 (d_i z/t)^{1/2} \, I_1(2 \, \beta_1 \sqrt{d_i z t}) \tag{3.6}$$

depends only on d_i and β_1, but not on $\beta_i \, (i \geq 2)$.

Furthermore, if one puts

$$a_r = \sum_1^r {}_i \, d_i \quad , \quad b_r = \sum_1^r {}_i \, B_i \quad , \quad \chi_r = \prod_i^r {}_i \, (\beta_i/\beta_1)^2$$

as for the convolution $\varphi_1 * \varphi_2$, by (3.5) one has

$$e^{-b_2 z} \, \varphi_1 * \varphi_2 < \chi_2 \, e^{-\beta_1 (t + a_2 z)} \, \lambda(d_1, t) * \lambda(d_2, t) \, . \tag{3.7}$$

Consequently, applying $(2.7)_2$-(3.5)-(3.6), one deduces

$$e^{-b_2 z} \, H_2 \leq \chi_2 \, e^{-\beta_1 (t + a_2 z)} \, [\lambda(d_1, t) + \lambda(d_2, t) + \lambda(d_1, t) * \lambda(d_2, t)] \, . \tag{3.8}$$

But, as one can prove by means of formulae of $\mathscr{L}_t$-transformation, it is

$$\lambda(d_i, t) * \lambda(d_j, t) + \lambda(d_i, t) + \lambda(d_j, t) = \lambda(d_i + d_j, t) \tag{3.9}$$

and so by (3.8)

$$e^{-b_2 z} \, H_2 \leq \chi_2 \, e^{-\beta_1 (t + a_2 z)} \, \lambda(d_1 + d_2, t) \, .$$

At this point it is easy to verify that the procedure can be iterated indefinitely. Observing that $g_n(0) = g_0 = b_n$, one deduces

$$e^{-g_0 z} \, H_n \leq \chi_n \, e^{-\beta_1 (t + a_n z)} \, \lambda(a_n, t) = \chi_n \, e^{-\beta_1 a_n z} \, H_1(\beta_1 \, ; \, \beta_1 \, a_n) \, ,$$

hence

$$A_n''(\beta_1 \ldots \beta_n; B_1 \ldots B_n) \leq \chi_n \, A_1''(\beta_1, \beta_1 \, a_n)$$

with A_i'' defined in (3.3). On the other hand, A' being a decreasing function of $g_n(0) = b_n \geq \beta_1 \, a_n$ and observing that $\chi_n > 1$, one has easily $A'(b_n) \leq \chi_n \, A'(\beta_1 \, a_n)$. Thus the proof is complete.

4. We will outline briefly some consequences of Theorem 0.1.

The prescribed past history of the stress $\sigma(x, \tau)$ (with $\tau \in (-\infty, 0[)$ by hypothesis (see [11]) is such that the integral in (1.1) is meaningful. Consequently it is

$$\lim_{t\to\infty} \int_{-\infty}^{0} g_n(t-\tau)\,\sigma(x,\tau)\,d\tau = 0 .$$

Therefore, according to well-known theorems on the asymptotic behavior of the convolutions [12] and $g_n(t)$ being summable on R^+, by (1.1) one has

$$\lim_{t\to\infty} e(x,t) = J_n(0)\left[1 + \int_0^{\infty} g_n(t)\,dt\right] \lim_{t\to\infty} \sigma(x,t) \tag{4.1}$$

provided that $\lim\limits_{t\to\infty} \sigma$ *exists.*

As (0.7) shows, the asymptotic relation (4.1) remains unaltered with $g_1(t)$ instead of $g_n(t)$. Further, by (1.2)-(1.4) it is

$$\frac{J_n(\infty)}{J_n(0)} = \frac{J_1(\infty)}{J_1(0)} = 1 + a_n \tag{4.2}$$

and so, in any case, $e(\infty) = J_n(\infty)\,\sigma(\infty)$ results.

These observations show that $\mathscr{B}_1$ is an asymptotic meaningful model and that when t is large the behaviour of $\mathscr{B}_n$ approaches that of an elastic material with modulus $J_n(\infty)$.

Theorem 0.1 and the explicit solution of the Cauchy problem established in [11] make rigorous these qualitative considerations.

To simplify, we will assume that the past history of the stress is negligible so that $\sigma(x,\tau) = 0$ for all $\tau \leq 0$. Then, by (1.1) one has $u(x,0) = 0$ and $\tilde{f}$ reduces itself (see [11]) to

$$f_* = \rho^{-1} f + \rho^{-1} \int_0^{t} g_n(t-\tau)\, f(x,\tau)\,d\tau , \tag{4.3}$$

where ρ is the density and f the body force.

Thus, referring to the half-space $Y^2_+ \equiv \{(x,t) : x \in R ,\, t > 0\}$, the initial value problem $\mathscr{P}_n$ for (0.1)-(0.2) is

$$L\,u = -f_* \qquad (x,t) \in Y_+ \tag{4.4}$$

$$u(x,0) = 0 \quad , \quad u_t(x,0) = f_1(x) \qquad x \in R . \tag{4.5}$$

Consider now the solution u_1 of the *simplest* problem $\mathcal{P}_1$ related to g_1

$$(4.6) \qquad L_1 u_1 = c^2 \partial_x u_1 - \partial_t^2 u_1 - \beta_1 a_n \int_0^t e^{-\beta_1(t-\tau)} \partial_\tau^2 u(x, \tau) \, d\tau = -\chi_n |f_*|$$

$$(4.7) \qquad u_1(x, 0) = 0 \quad , \quad \partial_t u_1(x, 0) = \chi_n |f_1(x)| .$$

According to [10], as the data $(|f^*|, |f_1|)$ are non-negative, one has $u_1(x, t) \geq 0$ everywhere in Y^2_+. Then, the following theorem holds.

THEOREM 4.1. *Let* u , u_1 *be the regular solutions of the problems* $\mathcal{P}_n$ *and* $\mathcal{P}_1$. *Then, everywhere in* Y^2_+ *it results*

$$(4.8) \qquad | u(x, t) | \leq u_1(x, t) .$$

Moreover, when the data (f_*, f_1) *have a same constant sign also* u *has the same sign.*

Proof. The proof is a consequence of Theorem 0.1 and of the formulae (3.3)-(3.4)-(3.5) of [11].

Now, as one can verify, (4.6) can be given the form

$$(4.7) \qquad \tau_1 \mu \partial_t (\partial_t^2 - c^2 \partial_x^2) u_1 + (\partial_t - c_0^2 \partial_x^2) u_1 = \mu \chi_n (\tau_1 \partial_t + 1) | f_* |$$

with

$$(4.8) \qquad \tau_1 = \beta_1^{-1} \quad , \quad c_0^2 = [\rho J_n(\infty)]^{-1} < c^2 \quad , \quad \mu = J_n(0)/J_n(\infty) .$$

The equation (4.7) points out clearly the meaningful parameters which together with c^2 affect *eventually* the wave behaviour of $\mathcal{B}_n$: the *characteristic time* of $\mathcal{B}_n$ given by the longest retardation time $\tau_1 = \beta_1^{-1}$ and the *characteristic speed* c_0 connected with the *regime value* $J_n(\infty)$ of the creep compliance.

Consequently, according to [10], we can state that when t is large compared with τ_1 $(t > \tau_1)$, the main signal is related to the speed c_0 connected with $J_n(\infty)$ and propagates in $\mathcal{B}_n$ as a diffusion process.

Rigorous estimates uniformly valid for all $t > 0$ can be evaluated by means of Theorem 6.3 of [10].

Finally, we remark that these results could be generalized to the case $n = \infty$.

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