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Geometria. — *A generalized exponential map for an affinely homogeneous cone.* Nota di UMBERTO SAMPIERI (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Dato un cono V aperto non vuoto, convesso, regolare e affinemente omogeneo in uno spazio vettoriale reale W di dimensione finita si prova che per ogni v appartenente a V esiste un diffeomorfismo $E_v : W \to V$ che soddisfa le condizioni seguenti E1) $E_v(0) = v$; E2) $\det(dE_v(y)) = \Phi_V(E_v(y))^{-1}$ per ogni y appartenente a W ove $\Phi_V : V \to \mathbf{R}^+$ è la funzione caratteristica di V.

INTRODUCTION

In the late 1950's Koecher ([6], [7]) proved that every affinely homogeneous self-adjoint cone is isomorphic to the set $\{\exp(a), a \in A\}$, where A is a compact Jordan algebra and exp is formally defined by the series $\sum_{n=0}^{\infty} a^n/n!$. (By cone we shall always mean a non-empty, open, convex, sharp cone in a finite dimensional real vector space W).

Some years later Vinberg, ([3], [4]) showed, as an easy consequence of his extension of the Morozov-Borel theorem to real Lie groups ([8]), that, given an affinely homogeneous cone $V \subseteq W$, there exists a triangular Lie group $\mathcal{T}$ acting simply and transitively on V. Therefore if τ is the Lie algebra of $\mathcal{T}$ and $\exp : \tau \to \mathcal{T}$ is the ordinary exponential map, the mapping $L_v : \mathcal{T} \to V$ defined by $L_v(y) = \exp(y)\, v$ is a diffeomorphism for all v in V.

In this paper we prove that, under the same hypotheses, for any v in V there exists a diffeomorphism $E_v : W \to V$ which satisfies the following conditions: E1) $E_v(0) = v$; E2) $\det(dE_v(y)) = \Phi_V(E_v(y))^{-1}$ for all $y \in W$ where $\Phi_V : V \to \mathbf{R}^+$ is the characteristic function of V.

Condition E2) seems to be the most natural multi-dimensional generalization of the ordinary differential equation $dE/dt = E$ characterizing the exponential function $E : \mathbf{R} \to \mathbf{R}^+$. Moreover, since the invariant measure on V is given by $d\mu_V = \Phi_V\, dm$ where dm is the Lebesgue measure on W, the linear mapping $\varepsilon_v : L^2(V, d\mu_V) \to L^2(W, dm)$ defined by $\varepsilon_v(f) = f \cdot E_v$ is a surjective isometry for any $v \in V$.

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Finally, as will be shown elsewhere, the diffeomorphisms E_v turn out to be a useful tool for the investigation of the relationship between Bergman weighted spaces and Hardy spaces on Siegel domains.

We avoid the whole algebraic machinery of T algebras, basing our construction instead on a fundamental paper by Koszul ([1]): principal results are summarized in section 1 to fix notations. In section 2 we develop the inductive construction of an affinely homogeneous cone ([3], [5]). This is carried out in a formulation needed in section 3 which is completely devoted to the construction of the diffeomorphisms E_v.

§ 1. HOMOGENEOUS REGULAR CONVEX DOMAINS IN AFFINE SPACES

Let F be a finite-dimensional real affine space and let Ω be a regular convex domain in F homogeneous under the action of the group, denoted henceforth by $\mathrm{Aff}(\Omega)$, of all affine transformations of F which map Ω bijectively onto itself.

Let G be a connected Lie subgroup of $\mathrm{Aff}(\Omega)$ acting transitively on Ω. If T is the vector space of translations on F, the action of G on Ω induces a linear representation $\tilde{f}$ of G on T. We shall denote by T_0 the subspace of T spanned by those traslations which are parallel to closed half-lines contained in Ω.

It is quite obvious that T_0 is $\tilde{f}$-stable, i.e. invariant by $\tilde{f}$. Under these hypotheses the following results due to Koszul ([1]) hold:

K1) Set $F_0 = (x_0 + t_0 \, , \, t_0 \in T_0)$ where x_0 is any point in F and let G_0 be the subgroup of G which fixes F_0. Then there exists a point $-p_0$ in F_0 fixed by G_0 and $\Omega_0 = \Omega \cap F_0$ is a cone with vertex in $-p_0$ on which G_0 acts transitively.

K2) There exists an $\tilde{f}$-stable subspace N_0 of T such that $T = T_0 \oplus N_0$ and there exists a bilinear symmetric form $\varphi : N_0 \times N_0 \to T_0$ which satifies:

i) $\tilde{f}(\lambda)_{|T_0} \varphi(u, v) = \varphi(\tilde{f}(\lambda)_{|N_0} u, \tilde{f}(\lambda)_{|N_0} v) \; \forall \lambda \in G_0, \forall (u, v) \in N_0 \times N_0$

ii) if $b \in F_0$ and $u \in N_0$, then $b + u \in \Omega$ iff $b - \varphi(u, u) \in \Omega_0$.

§ 2. INDUCTIVE CONSTRUCTION OF AN AFFINELY HOMOGENEOUS CONE

Let W be an $(n+1)$-dimensional real vector space and let V be an affinely homogeneous cone in W. The following result is due to Vinberg ([3]):

V1) There exists a basis on W and a triangular Lie group $\mathcal{T}$ with positive diagonal entries which acts simply transitively on V.

Henceforth we shall assume this basis chosen and identify W with $\mathbf{R}^{n+1}$. Points in W will be denote by $y = (x, s)$ where $x \in \mathbf{R}^n$ and $s \in \mathbf{R}$. We may suppose that $\Omega = \{x \in \mathbf{R}^n \text{ s.t. } (x, 1) \in V\}$ is not empty.

Let $A : \mathcal{T} \to T_+(n+1, \mathbf{R})$ be the linear representation of $\mathcal{T}$ given by:

$$A(\lambda) = \left[\begin{array}{c|c} \tilde{f}(\lambda) & \tilde{q}(\lambda) \\ \hline 0 & \tilde{r}(\lambda) \end{array}\right] \tag{2.1}$$

where $\tilde{f}(\lambda) \in T_+(n, \mathbf{R})$, $\tilde{q}(\lambda) \in \mathbf{R}^n$, $\tilde{r}(\lambda) > 0$ for all $\lambda \in \mathcal{T}$.

It is easy to prove that:

LEMMA 2.1. *Ω is a regular convex domain in $\mathbf{R}^n$.*

LEMMA 2.2. $G = \{(\tilde{f}(\lambda), \tilde{q}(\lambda)), \lambda \in \mathcal{T}, \tilde{r}(\lambda) = 1\}$ *is a connected Lie subgroup of* $\mathrm{Aff}(\Omega)$ *acting simply transitively on* Ω.

Let us fix a point $x_0 \in \Omega$. Then $y_0 = (x_0, 1) \in V$ and, since $\mathcal{T}$ acts simply transitively on V, we may define uniquely a diffeomorphism $\lambda : V \to \mathcal{T}$ such that

$$\lambda(y)\, y_0 = y \qquad \forall y \in V. \tag{2.2}$$

If we consider the C^∞ mapping $\gamma : V \to T_+(n+1, \mathbf{R})$

$$\gamma = A \circ \lambda \tag{2.3}$$

it follows from (2.1) and (2.2) that:

$$\tilde{r}(\lambda(y)) = \tilde{r}(\lambda(x, s)) = s \qquad \forall y = (x, s) \in V \tag{2.4}$$

Therefore setting:

$$\begin{cases} f : \to T_+(n, \mathbf{R}), f(x) = \tilde{f}(\lambda(x, 1)) \\ q : \to \mathbf{R}^n \qquad , q(x) = \tilde{q}(\lambda(x, 1)) \end{cases} \tag{2.5}$$

gives

LEMMA 2.3. $G = \{(f(x), q(x)), x \in \Omega\}$.

In accordance with section 1 set $F_0 = x_0 + T_0$. In view of K1), if G_0 is the subgroup of G which fixes F_0, there exists a point $-p_0$ in F_0 fixed by G_0 and $\Omega_0 = \Omega \cap F_0$ is a cone with vertex in $-p_0$ on which G_0 acts transitively.

LEMMA 2.4. $G_0 = \{(f(x), q(x)), x \in \Omega_0\}$.

Proof. Let $\sigma \in G_0$. Then by Lemma 2.3 $\sigma = (f(x), q(x))$, $x \in \Omega$. Moreover since F_0 is fixed by G_0 and $x_0 \in F_0$, $x = f(x)x_0 + q(x) = (f(x), q(x))x_0 \in F_0$ and so $x \in \Omega_0$. Conversely let $x \in \Omega_0$. Because G_0 acts transitively on Ω_0 there exists $z \in \Omega$ such that $(f(z), q(z)) \in G_0$ and carries x_0 to x. We conclude observing that $x = (f(z), q(z))x_0 = f(z)x_0 + q(z) = z$. ∎

Specializing K2) to this situation we many conclude:

LEMMA 2.5. *There exists an f-stable subspace* N_0 *of* $\mathbf{R}^n$ *such that* $\mathbf{R}^n = N_0 \oplus T_0$ *and there exists a bilinear symmetric form* $\varphi : N_0 \times N_0 \to T_0$ *which satisfies*

i) $f(x)_{|T_0}\varphi(u, v) = \varphi(f(x)_{|N_0}u, f(x)_{|N_0}v)\ \forall x \in \Omega_0, \forall (u, v) \in N_0 \times N_0$

ii) *if* $b \in F_0, u \in N_0$, *then* $b + u \in \Omega$ *iff* $b - \varphi(u, u) \in \Omega_0$.

Set $x_0 = x_0' + x_0''$, $x_0' \in T_0$ and $x_0'' \in N_0$. Given a vector $x \in \mathbf{R}^n$ we may write it uniquely as the sum of two vectors $t_0 \in T_0$ and $n_0 \in N_0$.

Since $x = (t_0 + x_0'' + n_0 - x_0'')$ condition ii) may be replaced by

ii′) $x \in \Omega$ if $\{t_0 - \varphi(n_0, n_0) + 2\varphi(n_0, x_0'') + x_0'' - \varphi(x_0'', x_0'')\} \in \Omega_0$.

Now let $x \in \mathbf{R}^n$, $s \in \mathbf{R}$ be such that $y = (x, s) \in V$. Since $s = \tilde{r}(\lambda(x, s))$ and $\mathcal{T}$ is connected we have $s > 0$. Therefore $(x, s) = (t_0 + n_0, s) \in V$ if and only if $(x/s, 1) \in V$ and $s > 0$ that is to say $s > 0$ and $(t_0/s + n_0/s) \in \Omega$. By ii′) we may conclude that $(x, s) \in V$ if and only if $s > 0$ and

$$\{t_0/s - \varphi(n_0, n_0)/s^2 + 2\varphi(n_0, x_0'')/s + x_0'' - \varphi(x_0'', x_0'')\} \in \Omega_0. \tag{2.6}$$

Recall that $V_0 = p_0 + \Omega_0$ is a cone in the vector space T_0. Thus if we set $l_0 = m_0 + p_0 = x_0'' - \varphi(x_0'', x_0'') + p_0$, (2.6) is equivalent to

$$\{st_0 - \varphi(n_0, n_0) + 2s\varphi(n_0, x_0'') + s^2 l_0\} \in V_0 \tag{2.7}$$

Summarizing we may state:

THEOREM 2.1. *Let* $\psi : \mathbf{R}^n \times \mathbf{R} \to T_0$ *defined by*:

$$\psi(x, s) = \psi(t_0 + n_0, s) = st_0 - \varphi(n_0, n_0) + 2s\varphi(n_0, x_0'') + s^2 l_0 \tag{2.8}$$

then $(x, s) \in V$ *if and only if* $s > 0$ *and* $\psi(x, s) \in V_0$.

LEMMA 2.6. *For all* $x \in \Omega_0$, $f(x)_{|T_0} \in \mathrm{Aut}(V_0)$ *and* $f(x)_{|T_0}(x_0 + p_0) = x + p_0$.

Proof. Let $z \in V_0$. Then $z - p_0 \in \Omega_0$ and, by Lemma 2.4

$$f(x)(z - p_0) + q(x) \in \Omega_0 . \tag{2.9}$$

Since $-p_0$ is fixed by G_0,

$$-f(x) p_0 + q(x) = -p_0 \tag{2.10}$$

and so, recalling (2.9)

$$f(x) z - p_0 \in \Omega_0 , \quad \text{i e. } f(x) z \in V_0 . \tag{2.11}$$

To conclude observe that by (2.9) we have

$$f(x)(x_0 + p_0) = f(x) x_0 + q(x) + p_0 = x + p_0 . \tag{2.12}$$

COROLLARY 2.1. *Let* $\Phi_{V_0} : V_0 \to \mathbf{R}^+$ *be the characteristic function of* V_0 *normalized by the condition*:

$$\Phi_{V_0}(x_0 + p_0) = 1 . \tag{2.13}$$

Then

$$\Phi_{V_0}(z) = \det f_{|T_0}^{-1}(z - p_0) \qquad \forall z \in V_0 . \tag{2.14}$$

Now let $x = t_0 + n_0 \in \Omega$ and set $f = f(\psi(x, 1) - p_0)$. Consider the vectors a, $c \in N_0$ and $b \in T_0$ defined by:

$$\begin{cases} a = n_0 - x_0'' \\ c = n_0 - f x_0'' \\ b = f l_0 - l_0 + \varphi(c, c) - 2\varphi(c, x_0'') . \end{cases} \tag{2.15}$$

LEMMA 2.8. *The linear mapping* $B : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n \times \mathbf{R}$

$$\begin{aligned} B(x', s') &= B(t_0' + n_0', s') = \\ &\{(f_{|T_0} t_0' + 2\varphi(f_{|N_0} n_0', a) + s' b) + (f_{|N_0} n_0' + s' c), s'\} \end{aligned} \tag{2.16}$$

belongs to Aut(V) *and satisfies*

$$B(x_0, 1) = (x, 1) . \tag{2.17}$$

Proof. Since

$$\det B = \det f_{|T_0}(\psi(x,1)-p_0)\,.\,\det f_{|N_0}(\psi(x,1)-p_0) = \det f(\psi(x,1)-p_0)\,. \tag{2.18}$$

B is not singular. A straightforward calculation shows that, in view of (2.15),

$$\psi(B(x',t')) = f_{|T_0}(\psi(x,1)-p_0)\,\psi(x',t') \tag{2.19}$$

and so by Lemma 2.6 and Theorem 2.1, $B(V) \subseteq V$.
Finally (2.17) is readily verified. ∎

COROLLARY 2.2. *Let* $\Phi_V : V \to \mathbf{R}^+$ *be the characteristic function of* V *normalized by the condition*

$$\Phi_V(x_0, 1) = 1\,. \tag{2.20}$$

Then for any $x \in \Omega$,

$$\Phi_V(x,1) = \det f^{-1}_{|T_0}(\psi(x,1)-p_0)\,.\,\det f^{-1}_{|N_0}(\psi(x,1)-p_0) = \Phi_{V_0}(\psi(x,1))\,.\,\det f^{-1}_{|N_0}(\psi(x,1)-p_0)\,. \tag{2.21}$$

Proof. This follows from Lemma 2.8, (2.18) and Corollary 2.1. ∎

§ 3. INDUCTIVE CONSTRUCTION OF A GENERALIZED EXPONENTIAL MAP FOR AN AFFINELY HOMOGENEOUS CONE

LEMMA 3.1. *Let* $M = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$ *be a square* $(l+m)$*-matrix where* A *is an* $(l \times l)$ *matrix and* B *is an* $(m \times m)$ *non-singular matrix. Then*

$$\det M = \det(A - CB^{-1}D)\cdot\det B\,. \tag{3.1}$$

Proof. Let $\Lambda = \{\lambda^k_\beta\}$ *be an* $(m \times l)$ matrix and set

$$M(\Lambda) = \begin{bmatrix} A(\Lambda) & C \\ D(\Lambda) & B \end{bmatrix} \tag{3.2}$$

where

$$A(\Lambda)^\alpha_\beta = A^\alpha_\beta + \sum_{k=1}^{m} \lambda^k_\beta C^\alpha_k \quad ; \quad D(\Lambda)^i_\beta = D^i_\beta + \sum_{k=1}^{m} \lambda^k_\beta B^i_k\,.$$

Since $M(\Lambda)$ is obtained from M by adding to the first l columns scalar multiples of the last m columns, we have $\det M(\Lambda) = \det M$ for all Λ. In particular if $\Lambda = -B^{-1}D$ it is easy to check that $A(\Lambda) = A - CB^{-1}D$ and $D(\Lambda) = 0$ and so (3.1) is proved. ∎

Now let us suppose that there exists a diffeomorphism $E_0 : T_0 \to V_0$ such that:

$$\begin{cases} E_0(0) = x_0 + p_0 \\ \det(dE_0(t_0)) = \Phi_{V_0}(E_0(t_0))^{-1} \qquad \forall t_0 \in T_0 \end{cases} \tag{3.3}$$

and let D_0 be its inverse.

We define a C^∞ mapping $E : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n \times \mathbf{R}$ by

$$\begin{aligned} E(t_0 + n_0, s) = e^s \{ & (E_0(t_0) + f_{|T_0}(E_0(t_0) - p_0)\, \varphi(n_0, n_0) + \\ & -(x_0'' + p_0)) + (f_{|N_0}(E_0(t_0) - p_0)\, n_0 + x_0''), 1\}. \end{aligned} \tag{3.4}$$

LEMMA 3.2. *E is a diffeomorphism of* $\mathbf{R}^n \times \mathbf{R}$ *onto* V *satisfying*

$$\begin{cases} E(0) = y_0 = (x_0, 1) \\ \det(dE(y)) = \Phi_V(E(y))^{-1}\ \forall y \in \mathbf{R}^n \times \mathbf{R}. \end{cases} \tag{3.5}$$

Proof. Observe that

$$E(t_0 + n_0, s) = e^s E(t_0 + n_0, 0) \tag{3.6}$$

and let us prove that

$$\psi(E(t_0 + n_0, s)) = e^{2s} E_0(t_0). \tag{3.7}$$

Indeed we have:

$$\begin{aligned} \psi(E(t_0 + n_0, s)) &= \psi(e^s E(t_0 + n_0, 0)) = e^{2s} \psi(E(t_0 + n_0, 0)) = \\ &= e^{2s} \{E_0(t_0) + f_{|T_0}(E_0(t_0) - p_0)\, \varphi(n_0, n_0) - x_0'' - p_0 + \\ &\quad - \varphi(f_{|N_0}(E_0(t_0) - p_0)\, u_0 + x_0'', f_{|N_0}(E_0(t_0) - p_0)\, n_0 + x_0'') + \\ &\quad + 2\varphi(f_{|N_0}(E_0(t_0) - p_0)\, n_0 + x_0'', x_0'') + p_0 + m_0\} = \\ &= e^{2s} \{E_0(t_0) + f_{|T_0}(E_0(t_0) - p_0)\, \varphi(n_0, n_0) - x_0'' - p_0 + \\ &\quad - f_{|T_0}(E_0(t_0) - p_0)\, \varphi(n_0, n_0) - \varphi(x_0'', x_0'') - 2\varphi(f_{|N_0}(E_0(t_0) - p_0)\, n_0, x_0'') + \\ &\quad + 2\varphi(f_{|N_0}(E_0(t_0) - p_0)\, n_0, x_0'') + 2\varphi(x_0'', x_0'') + \\ &\quad + p_0 + x_0'' - \varphi(x_0'', x_0'')\} = e^{2s} E_0(t_0). \end{aligned}$$

The identity (3.7) shows that $E(\mathbf{R}^n \times \mathbf{R}) \subseteq V$. Now let $(t_0' + n_0', s') \in V$. It follows from Theorem 2.1 that $s' > 0$ and $\psi(t_0' + n_0', s') \in V_0$. Therefore $E(t_0 + n_0, s) = (t_0' + n_0', s')$ if and only if

$$
(3.8) \quad \begin{cases} e^s = s' \quad \text{iff} \quad s = \ln s' \\ \psi(E(t_0 + n_0, s)) = e^{2s} E_0(t_0) = (s')^2 E_0(t_0) = \psi(t_0' + n_0', s') \quad \text{iff} \\ t_0 = D_0\left(\psi\left(\dfrac{t_0' + n_0'}{s'}, 1\right)\right) \\ e^s (f_{|N_0}(E_0(t_0) - p_0) n_0 + x_0'') = n_0' \quad \text{iff} \quad n_0 = f^{-1}\left(\psi\left(\dfrac{t_0' + n_0'}{s'}, 1\right) + \right. \\ \left. - p_0\right)\left(\dfrac{n_0'}{s'} - x_0''\right) \end{cases}
$$

so we may conclude that $E : \mathbf{R}^n \times \mathbf{R} \to V$ is a diffeomorphism and its inverse D is given by:

$$
(3.9) \quad D(t_0' + n_0', s') = \left\{D_0\left(\psi\left(\frac{t_0' + n_0'}{s'}, 1\right)\right) + f^{-1}\left(\psi\left(\frac{t_0' + n_0'}{s'}\right) - p_0\right)\right. \\ \left.\left(\frac{n_0'}{s'} - x_0''\right), \quad \ln s'\right\}.
$$

Recalling (3.4) it is easy to check that

$$
(3.10) \qquad E(0) = \{E_0(0) - x_0'' - p_0 + x_0'', 1\} = \{x_0 + p_0 - p_0, 1\} = y_0 .
$$

To evaluate $\det(dE(y))$, fix bases $(t_1, \dots, t_l)$ for T_0 and $(n_1, \dots, n_m)$ for N_0. Indices from 1 to m will be denoted by Latin letters and indices from 1 to l by Greek ones. To simplify notation set

$$
f_{|T_0}(E_0(t_0) - p_0) = \{f_\beta^\alpha\} \qquad \text{and} \quad f_{|N_0}(E_0(t_0) - p_0) = \{f_k^i\}
$$

and rewrite (3.4) as

$$
(3.11) \qquad E(t_0 + n_0, s) = \{E^\alpha(t_0 + n_0, s) \quad , \quad E^i(t_0 + n_0, s) \quad , \quad e^s\}
$$

where

$$
\begin{cases} E^\alpha(t_0 + n_0, s) = e^s \{E_0^\alpha(t_0) + f_\gamma^\alpha \varphi_{ik}^\gamma n_0^i n_0^k - (x_0'' + p_0)^\alpha\} \\ E^i(t_0 + n_0, s) = e^s \{f_j^i n_0^j + (x_0'')^i\} . \end{cases}
$$

Then

$$\text{(3.12)} \qquad dE(t_0 + n_0, s) = e^s \left[\begin{array}{cc|c} A & C & \\ & & * \\ D & B & \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right]$$

where

$$\begin{cases} A^\alpha_\beta = dE_0(t_0)^\alpha_\beta + \dfrac{\partial f^\alpha_\gamma}{\partial t^\mu_0} dE_0(t_0)^\mu_\beta \varphi^\gamma_{ik} n^i_0 n^k_0 \\ C^\alpha_j = 2 f^\alpha_\gamma \varphi^\gamma_{jk} n^k_0 \\ D^k_\alpha = \dfrac{\partial f^k_j}{\partial t^\mu_0} dE_0(t_0)^\mu_\alpha n^j_0 \\ B^i_k = f^i_k . \end{cases}$$

Since B is not singular, in view of Lemma 3.1 we may conclude:

$$\text{(3.13)} \qquad \det(dE(t_0 + n_0, s_0)) = e^{s(n+1)} \det(A - CB^{-1} D) \det B .$$

Now:

$$(CB^{-1} D)^\alpha_\beta = C^\alpha_j (B^{-1})^j_k D^k_\beta = 2 f^\alpha_\gamma \varphi^\gamma_{ji} (f^{-1})^j_k \frac{\partial f^k_\rho}{\partial t^\mu_0} dE_0(t_0)^\mu_\beta n^\rho_0 n^i_0 =$$

$$= 2 \left\{ f_{|T_0} \varphi \left((f^{-1})^j_k \frac{\partial f^k_\rho}{\partial t^\mu_0} dE_0(t_0)^\mu_\beta n^\rho_0 , n^j_0 \right) \right\}^\alpha = 2 \left\{ \varphi \left(\frac{\partial f^m_\rho}{\partial t^\mu_0} dE_0(t_0)^\mu_\beta n^\rho_0 , f^m_j n^j_0 \right) \right\}^\alpha =$$

$$= 2 f^\rho_j \frac{\partial f^m_k}{\partial t^\mu_0} dE_0(t_0)^\mu_\beta \varphi^\alpha_{m\rho} n^j_0 n^k_0 .$$

But, since $E_0(t_0) - p_0 \in \Omega_0$

$$\text{(3.14)} \qquad f^\alpha_\gamma \varphi^\gamma_{jk} n^j_0 n^k_0 = \varphi^\alpha_{m\rho} f^m_j f^\rho_k n^j_0 n^k_0$$

and taking derivatives with respect to $\dfrac{\partial}{\partial t^\beta_0}$,

$$\text{(3.15)} \qquad \frac{\partial f^\alpha_\gamma}{\partial t^\mu_0} dE_0(t_0)^\mu_\beta \varphi^\gamma_{jk} n^j_0 n^k_0 = 2 \varphi^\alpha_{m\rho} \frac{\partial f^m_j}{\partial t^\mu_0} f^\rho_k dE_0(t_0)^\mu_\beta n^j_0 n^k_0 .$$

Finally:

$$\text{(3.16)} \qquad A - CB^{-1} D = dE_0(t_0) .$$

Substituting (3.16) into (3.13) gives

$$\det dE(t_0 + n_0, s) = e^{s(n+1)} \det dE_0(t_0) \det f_{|N_0}(E_0(t_0) - p_0) \tag{3.17}$$

and, from (3.7),

$$\det dE(t_0 + n_0, s) = e^{s(n+1)} \Phi_{V_0}(\psi(E(t_0 + n_0, 0))^{-1} \det f_{|N_0}(\psi(E(t_0 + n_0, 0) \to p_0) . \tag{3.18}$$

In view of Corollary 2.2 we may conclude that:

$$\begin{aligned}\det dE(t_0 + n_0, s) &= e^{s(n+1)} \Phi_V(E(t_0 + n_0, 0))^{-1} = \\ &= \Phi_V^{-1}(e^s E(t_0 + n_0, 0))^{-1} = \Phi_V^{-1}(E(t_0 + n_0, s))^{-1} .\end{aligned} \tag{3.19}$$ ∎

Observe that if $v \in V$ and $A_v \in \mathrm{Aut}(V)$ is such that $A_v(y_0) = v$, then the diffeomorphism $E_v : \mathbf{R}^{n+1} \to V$ defined by

$$E_v = A_v \circ E \tag{3.20}$$

satisfies

$$\begin{cases} E_v(0) = A_v(E(0)) = A_v(y_0) = v \\ \det dE_v(y) = \det A_v \det dE(y) = \det A_v \Phi_V(E(y))^{-1} = \\ = \Phi_V(E_v(y))^{-1} \; \forall y \in \mathbf{R}^{n+1} . \end{cases} \tag{3.21}$$

We are now in a position to prove:

THEOREM 3.1. *Let* V *be an affinely homogeneous cone in a finite-dimensiona real vector space* W. *Then for all* $v \in V$ *there exists a diffeomorphism* $E_v : W \to V$ *such that*:

E1) $E_v(0) = v$

E2) $\det(dE_v(y)) = \Phi_v(E_v(y))^{-1}$ for all $y \in W$.

Proof. By induction on the dimension of W.

If $\dim(W) = 1$ we may assume $W = \mathbf{R}$, $V = \mathbf{R}^+$, $\Phi_V(t) = 1/t$. Then $E_v(s) = ve^s$ trivially satisfies conditions E1) and E2). Suppose now that the theorem is proved for all $m \leq n$ and let $\dim(W) = n + 1$. By the inductive hypothesis there exists $E_0 : T_0 \to V_0$ such that (3.3) holds. Then, by Lemma 3.2, E satisfies (3.5) and, consequently, E_v, defined by (3.20) satisfies conditions E1) and E2). ∎

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