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# Rendiconti

Eugenio Sinestrari

## Classical solutions of parabolic equations in Hölder spaces

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# RENDICONTI

DELLE SEDUTE

## DELLA ACCADEMIA NAZIONALE DEI LINCEI

**Classe di Scienze fisiche, matematiche e naturali**

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*Presiede il Presidente della Classe* GIUSEPPE MONTALENTI

## SEZIONE I

**(Matematica, meccanica, astronomia, geodesia e geofisica)**

**Analisi matematica.** — *Classical solutions of parabolic equations in Hölder spaces* (*). Nota di EUGENIO SINESTRARI (**), presentata (***) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Sono dati nuovi teoremi di esistenza per soluzioni regolari di equazioni di evoluzione paraboliche astratte con applicazioni all'equazione del calore in spazi di funzioni holderiane e alle equazioni semilineari.

### 1. INTRODUCTION

In this note we wish to investigate existence, uniqueness and regularity properties of the solutions of the linear abstract Cauchy problem

$$(1.1)\qquad \begin{cases} u'(t) = Au(t) + f(t) & 0 < t \leq T \\ u(0) = x \end{cases}$$

where $A \colon D_A \subset E \to E$ is a linear closed operator in the Banach space E (with norm $\|\,.\,\|$) and $f \in C(0, T; E)$ i.e. $f$ is a continuous function from $[0, T]$ into

(*) Work done as a member of G.N.A.F.A. of the C.N.R.

(**) Istituto Matematico Castelnuovo, Università di Roma, P. Aldo Moro 7 - 00185 Roma.

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E. $D_A$ will be always endowed with the graph norm. In [6] we proved various results about this problem under the assumption that A generates a bounded analytic semigroup; in particular that $\overline{D}_A = E$. Here we want to drop this condition in view of the applications which will be given in the last section, for example to the classical heat equation.

More precisely we will assume that

$$(1.2)\quad \begin{cases} \text{there exist } \theta \in ]\pi/2, \pi[ \text{ and } M > 0 \text{ such that if } z \in \mathbf{C} - \{0\} \\ \text{and } |\arg z| < \theta \text{ then } (z - A)^{-1} \in \mathscr{L}(E) \text{ and } \| z(z-A)^{-1} \|_{\mathscr{L}(E)} \leq M . \end{cases}$$

It can be seen through the usual Dunford integral (see [2] formula (1.50) of page 487) that one can define a bounded analytic semigroup $\exp(At)$ which however is not strongly continuous at $t = 0$.

DEFINITION 1.1. *A function* $u \in C^1(0, T; E) \cap C(0, T; D_A)$ *such that* (1.1) *holds for* $0 \leq t \leq T$ *is called a* strict solution *of* (1.1).

DEFINITION 1.2. *A function* $u \in C(0, T; E) \cap C^1(0^+, T; E) \cap C(0^+, T; D_A)$ *such that* (1.1) *holds for* $0 < t \leq T$ *is called a* classical solution *of* (1.1).

Here $C^1(0^+, T; E)$ is the class of functions $u \in C^1(\varepsilon, T; E)$ for each $\varepsilon \in ]0, T]$ and similarly for $C(0^+, T; D_A)$. It can be easily proved that the classical (and hence the strict) solution is unique (if it exists) because it is given by the formula

$$(1.3)\qquad u(t) = \exp(At)\, x + \int_0^t \exp(A(t-s))\, f(s)\, ds .$$

DEFINITION 1.3. *Given* $x \in D_A$ *and* $f \in C(0, T; E)$, *the right-hand side of* (1.3) *is called the* mild solution *of* (1.1).

It is known that in general a mild solution is not classical. It is also easy to show that if a classical solution $u$ exists then $u'(t) \in \overline{D}_A$ for $0 < t \leq T$ and if $u$ is a strict solution then $x \in D_A$ and $Ax + f(0) \in \overline{D}_A$.

We will introduce now some intermediate spaces between $D_A$ and E (see [1]).

DEFINITION 1.4. *For each* $\theta \in ]0, 1[$ *let us define*

$$D_A(\theta, \infty) = \{x \in E, \| x \|_\theta = \sup_{t>0} \| t^{1-\theta} A \exp(At)\, x \| < \infty\}$$

*with norm*

$$\| x \|_{D_A(\theta,\infty)} = \| x \| + \| x \|_\theta .$$

We shall consider also the subspace of $D_A(\theta, \infty)$ defined as

DEFINITION 1.5. $D_A(\theta) = \{x \in E, \lim_{t\to 0} t^{1-\theta} A \exp(At)x = 0\}$, *with the norm of* $D_A(\theta, \infty)$.

If $0 < \theta_1 < \theta_2 < 1$ we have $D_A \subset D_A(\theta_2, \infty) \subset D_A(\theta_1)$.

Let us also recall the definition of hölder and little hölder continuous functions' spaces from $[0, T]$ into E:

DEFINITION 1.6.

$$C^\theta(0, T; E) = \left\{u : [0, T], \to E, \| u \|_\theta = \sup_{t,s\in[0,T]} \frac{\| u(t) - u(s) \|}{| t - s |^\theta} < \infty\right\}$$

*with norm* $\| u \|_{C^\theta(0,T;E)} = \| u \|_{C(0,T;E)} + \| u \|_\theta$.

The same definition with $\theta = 1$ gives the Lipschitz continuous functions, space which will be denoted by Lip $(0, T; E)$

DEFINITION 1.7.

$$h^\theta(0, T; E) = \left\{u : [0, T] \to E, \lim_{h\to 0} \sup_{|t-s|\leq h} \frac{\| u(t) - u(s) \|}{| t - s |^\theta} = 0\right\}$$

*with the norm of* $C^\theta(0, T; E)$ *is the little hölder continuous functions' space* (see [6]

DEFINITION 1.8. $B(0, T; E) = \{u: [0, T] \to E, \sup_{0\leq t\leq T} \| u(t) \| < \infty\}$ *with the sup-norm.*

Now we give some estimates of the behaviour near $t = 0$ or $t = \infty$ of the operator valued function $t \to A^n \exp(At)$, $n = 0, 1, \ldots,$ when $A^n \exp(At)$ is considered as an element of $\mathscr{L}(E_1, E_2)$ with $E_i = E$ or $D_A(\theta, \infty)$.

THEOREM 1.9. *Let us set* $D_A(0, \infty) = E$ *and*

$\| \cdot \|_{\mathscr{L}(D_A(\theta_1,\infty), D_A(\theta_2,\infty))} = \| \cdot \|_{\theta_1,\theta_2}$, *for* $\theta_1, \theta_2 \in [0, 1[$. *For fixed* $t > 0$, $n \in N - \{0\}$ *and* $0 < \alpha < \theta < \beta < 1$ *we have*

$$\| \exp(At) \|_{\theta,0} \leq M_0$$

$$\| A^n \exp(At) \|_{\theta,0} \leq M_{n,\theta}\, t^{\theta-n} \quad \textit{or} \leq M_n\, t^{-n}$$

$$\| \exp(At) \|_{0,\theta} \leq M_0 + M_1\, t^{-\theta}$$

$$\| \exp(At) \|_{\alpha,\theta} \leq \max(M_0, t^{\alpha-\theta})$$

$$\| A^n \exp(At) \|_{0,\theta} \leq M_n\, t^{-n} + M_{n+1}\, t^{-n-\theta}$$

$$\| A^n \exp(At) \|_{\alpha,\theta} \leq M_{n+1,\alpha} \cdot \max(t^{-n}, t^{-n-\theta+\alpha})$$

$$\| \exp(At) \|_{\beta,\theta} \leq M_0 (1 + M_1)$$

$$\| A^n \exp(At) \|_{\beta,\theta} \leq M'_{n,\beta}\, t^{\beta-n} + M_{n,\beta,\theta}\, t^{\beta-n-\theta} , \quad \text{or} \leq M_n (1 + M_1) t^{-n}$$

*where*

$$M_k = \sup_{t>0} \| t^k A^k \exp(At) \|_{\mathscr{L}(E)} , \qquad k = 0, 1, \ldots$$

$$M_{n,\theta} = n^{n-\theta} (n-1)^{1-n} M_{n-1} , n > 1 ; M_{n,1} = 1$$

$$M'_{n,\beta} = 2^{n-\beta} M_0 M_{n,\beta}$$

$$M_{n,\beta,\theta} = 2^{n+\theta-\beta} M_1 M_{n,\beta} .$$

*Remark.* In the previous theorem, when two estimates are given, the first one is better for $t \to 0$, whereas the second one is better for $t \to \infty$.

## 2. Mild solutions

The following embedding theorems can be proved with the aid of the preceding definitions

THEOREM 2.1. *For* $T > 0$ *we have*

1) $C^{\gamma}(0, T; E) \cap B(0, T; D_A(\gamma, \infty)) \subsetneq C^{\varepsilon}(0, T; D_A(\gamma - \varepsilon)) , 0 < \varepsilon < \gamma < 1$
2) $h^{\gamma}(0, T; E) \cap C(0, T; D_A(\gamma, \infty)) \subsetneq h^{\varepsilon}(0, T; D_A(\gamma - \varepsilon)) , 0 < \varepsilon < \gamma < 1$
3) $\text{Lip}(0, T; E) \cap B(0, T; D_A) \subsetneq C^{1-\theta}(0, T; D_A(\theta)) \qquad , 0 < \theta < 1$
4) $C^1(0, T; E) \cap C(0, T; D_A) \subsetneq h^{1-\theta}(0, T; D_A(\theta)) \qquad , 0 < \theta < 1$
5) $C(0, T; E) \cap B(0, T; D_A(\theta, \infty)) \subsetneq C(0, T; D_A(\theta')) \quad , 0 < \theta' < \theta < 1 .$

By using these results and the properties of the intermediate spaces (see [7]) we can study the two terms appearing in (1.3).

LEMMA 2.2. *Setting*

$$u_0(t) = \exp(At)\, x \tag{2.1}$$

*we have for each* $T > 0$ :

1) $x \in \overline{D}_A \Rightarrow u_0 \in C(0, T; E)$ *and* $u_0$ *is a classical solution of* (1.1) *with* $f = 0$;
2) $x \in D_A(\gamma, \infty) \Rightarrow u_0 \in C^{\gamma}(0, T; E) \cap B(0, T; D_A(\gamma, \infty)) \cap C^{\varepsilon}(0, T; D_A(\gamma - \varepsilon))$ *for each* $\varepsilon \in ]0, \gamma[$.
3) $x \in D_A(\gamma) \Rightarrow u_0 \in h^{\gamma}(0, T; E) \cap C(0, T; D_A(\gamma)) \cap h^{\varepsilon}(0, T; D_A(\gamma - \varepsilon))$ *for each* $\varepsilon \in ]0, \gamma[$.

4) $x \in D_A \Rightarrow u_0 \in \mathrm{Lip}(0, T; E) \cap B(0, T; D_A) \cap C^{1-\theta}(0, T; D_A(\theta))$, *for each* $\theta \in ]0, 1[$.

5) $x \in D_A$, $Ax \in \bar{D}_A \Rightarrow u_0 \in C^1(0, T; E) \cap C(0, T; D_A) \cap h^{1-\theta}(0, T; D_A(\theta))$, *for each* $\theta \in ]0, 1[$ *and* $u_0$ *is a strict solution of* (1.1) *with* $f = 0$.

The next lemma gives a generalization of Theorems 1, 4, 5 of [6]; this is useful in the application to semilinear equations (see section 4).

LEMMA 2.3. *Given* $f \in C(0, T; E)$ *and set*

$$u_1(t) = (\exp(A) * f)(t) = \int_0^t \exp(A(t-s)) f(s)\, ds \tag{2.2}$$

*we have*

$$u_1 \in C^{1-\theta}(0, T; D_A(\theta)), \quad \textit{for each } \theta \in ]0, 1[ \tag{2.3}$$

*If in addition* $f \in C(0, T; D_A)$ *then*

$$u_1 \in h^{1-\theta}(0, T; D_A(\theta)), \quad \textit{for each } \theta \in ]0, 1[. \tag{2.4}$$

With the aid of the preceding lemmas we can prove a regularity property of the mild solution.

THEOREM 2.4. *If* $f \in C(0, T; E)$ *and* $x \in \bar{D}_A$ *the mild solution* $u$ *of* (1.1) *belongs to* $C(0, T; E) \cap C^{1-\theta}(0+, T; D_A(\theta))$ *for each* $\theta \in ]0, 1[$. *If in addition* $f \in C(0, T; \bar{D}_A)$ *then* $u \in h^{1-\theta}(0+, T; D_A(\theta))$ *for each* $\theta \in ]0, 1[$.

## 3. CLASSICAL AND STRICT SOLUTIONS

We now give conditions of $f$ and $x$ to obtain classical solutions of (1.1). Let us first examine the case $f(0) = x = 0$.

LEMMA 3.1. *If* $f \in C^\beta(0, T; E)$ *with* $0 < \beta < 1$ *and* $f(0) = 0$ *then* $u_1$ (*given by* (2.2)) *is a strict solution of* (1.1) *with* $x = 0$ *and moreover*

$$u_1', \; Au_1 \in C^\beta(0, T; E) \tag{3.1}$$

$$u_1' \in B(0, T; D_A(\beta, \infty)). \tag{3.2}$$

*If* $f \in h^\beta(0, T; E)$ *and* $f(0) = 0$ *we also have*

$$u', \; Au_1 \in h^\beta(0, T; E) \tag{3.3}$$

$$u_1' \in C(0, T; D_A(\beta)). \tag{3.4}$$

Properties (3.1) and (3.3) were proved in [6]. The others are useful to find results of Schauder type for the solutions of the Cauchy-Dirichlet problem for the classical heat equation (see sect. 4). Now we can solve (1.1) with the method used in sect. 4 of [6] and with the aid of Lemma 2.2.

THEOREM 3.2. *Let* $f \in C^{\beta}(0, T; E), 0 < \beta < 1$:

1) *if* $x \in \overline{D}_A$ *then* (1.1) *has a classical solution u such that*

$$u' \in C^{\beta}(0^+, T; E) \cap B(0^+, T; D_A(\beta, \infty)) \cap C^{\varepsilon}(0^+, T; D_A(\beta - \varepsilon), \quad \forall \varepsilon \cdot \in ]0, \beta[ . \tag{3.5}$$

$$Au \in C^{\beta}(0^+, T; E) \tag{3.6}$$

2) *if* $x \in D_A$ *and* $Ax + f(0) \in \overline{D}_A$ *then u is a strict solution*

3) *if* $x \in D_A$ *and* $Ax + f(0) \in D_A(\beta, \infty)$, *then properties* (3.5) *and* (3.6) *hold with* $(0^+, T)$ *replaced by* $(0, T)$.
*Moreover let* $f \in h^{\beta}(0, T; E)$

4) *if* $x \in \overline{D}_A$ *then for the classical solution u we have that*

$$u' \in h^{\beta}(0^+, T; E) \cap C(0^+, T; D_A(\beta)) \cap h^{\varepsilon}(0^+, T; D_A(\beta - \varepsilon)), \quad \forall \varepsilon \in ]0, \beta[ \tag{3.7}$$

$$Au \in h^{\beta}(0^+, T; E) \tag{3.8}$$

5) *if* $x \in D_A$ *and* $Ax + f(0) \in D_A(\beta)$ *then properties* (3.7) *and* (3.8) *hold with* $(0^+, T)$ *replaced by* $(0, T)$.

With references to the physical applications of (1.1) (see sect. 4) the preceding theorem can be considered as a time-regularity result while the next lemma refers to the space-regularity which will be studied firstly when $x = 0$.

LEMMA 3.3. *Let* $f \in C(0, T; E) \cap B(0, T; D_A(\theta, \infty))$ *with* $0 < \theta < 1$. *Then* $u_1$ *is a strict solution of* (1.1) *with* $x = 0$ *and we have*

$$u_1', Au_1 \in B(0, T; D_A(\theta, \infty)) \tag{3.9}$$

$$Au_1 \in C^{\theta}(0, T; E). \tag{3.10}$$

*If in addition* $f \in C(0, T; D_A(\theta))$, *then*

$$u_1', Au_1 \in C(0, T; D_A(\theta)) \tag{3.11}$$

$$Au_1 \in h^{\theta}(0, T; E). \tag{3.12}$$

Now by using Lemma 2.2 we can prove the following

THEOREM 3.4. *Let* $f \in C(0, T; E) \cap B(0, T; D_A(\theta, \infty))$, $0 < \theta < 1$;

1) *if* $x \in \overline{D}_A$ *then problem* (1.1) *has a classical solution* $u$ *verifying*

$$u' \in B(0^+, T; D_A(\theta, \infty)) \tag{3.13}$$

$$Au \in C^\theta(0^+, T; E) \cap B(0^+, T; D_A(\theta, \infty)) \cap C^\varepsilon(0^+, T; D_A(\theta - \varepsilon)), \quad \forall \varepsilon \in ]0, \theta[ \tag{3.14}$$

2) *if* $x \in D_A$ *and* $Ax \in \overline{D}_A$ *then the solution is strict*

3) *if* $x \in D_A$ *and* $Ax \in D_A(\theta, \infty)$ *then properties* (3.13), (3.14) *hold with* $(0^+, T)$ *replaced by* $(0, T)$.
*Moreover, let* $f \in C(0, T; D_A(\theta))$

4) *if* $x \in \overline{D}_A$, *then for the classical solution* $u$ *we have that*

$$u' \in C(0^+, T; D_A(\theta)) \tag{3.15}$$

$$Au \in h^\theta(0^+, T; E) \cap C(0^+, T; D_A(\theta)) \cap h^\varepsilon(0^+, T; D_A(\theta - \varepsilon)), \quad \forall \varepsilon \in ]0, \theta[ \tag{3.16}$$

5) *if* $x \in D_A$ *and* $Ax \in D_A(\theta)$ *then properties* (3.15), (3.16) *hold with* $(0^+, T)$ *replaced by* $(0, T)$.

## 4. APPLICATIONS

We want to apply our abstract results to the study of the strict solutions of the initial boundary value problem of the classical heat equation (with Dirichlet boundary conditions)

$$\begin{cases} v_t(t, x) = \Delta v(t, x) + g(t, x) & 0 \leq t \leq T, x \in \overline{\Omega} \\ v(t, x) = 0 \quad , \quad 0 \leq t \leq T, x \in \partial\Omega \\ v(t, x) = \varphi(x) \quad , \quad x \in \overline{\Omega}. \end{cases} \tag{4.1}$$

Here $\Omega \subset R^n$ is an open bounded set with 'smooth' boundary $\partial\Omega$ and $g, \varphi$ are given functions. We shall solve (4.1) under assumptions which are usual in hölder continuous functions' spaces (see [3], Th. 5.2, p. 320).

THEOREM 4.1. *Let* $g \in C([0, T] \times \overline{\Omega}$ *be such that* $g(., x) \in C^{\theta/2}(0, T)$ *for some* $\theta \in ]0, 1[$, *uniformly for* $x \in \overline{\Omega}$ *and* $g(t, .) \in C^\theta(\overline{\Omega})$ *uniformly for* $t \in [0, T]$

*and let* $\varphi \in C^{2+\theta}(\bar{\Omega})$ *verify the compatibility conditions*

$$\varphi(x) = \Delta\varphi(x) + g(0, x) = 0 \quad , \quad x \in \partial\Omega$$

*Then there exists a unique function* $v$ *such that*

1) $v, v_t, \Delta v \in C([0, T] \times \Omega)$

2) $v_t(., x), \Delta v(., x) \in C^{\theta/2}(0, T)$ *uniformly for* $x \in \bar{\Omega}$ *and*
$v_t(t, .), \Delta v(t, .) \in C^{\theta}(\bar{\Omega})$ *uniformly for* $t \in [0, T]$

3) $v(t, .) \in C^{2+\theta}(\bar{\Omega})$ *uniformly for* $t \in [0, T]$

4) (4.1) *are verified in* $[0, T] \times \bar{\Omega}$.

*Proof.* We can consider (1.1) as the abstract version of (4.1) if we set $E = C(\bar{\Omega})$ with the sup-norm, $D_A = \{w \in C(\Omega), \Delta w \in C(\bar{\Omega}), w = 0 \text{ on } \partial\Omega\}$ (here $\Delta$ is the Laplacian in the sense of distributions on $\Omega$) and $Aw = \Delta w$ and if we define $u, f: [0, T] \to E$ as $u(t)(x) = v(t, x)$ and $f(t)(x) = g(t, x)$ for $t \in [0, T]$ and $x \in \bar{\Omega}$. By virtue of [8] we know that A verifies condition (1.2); it is also proved in [4] that we have $D_A(\theta, \infty) \simeq \{w \in C^{2\theta}(\bar{\Omega}), w = 0 \text{ on } \partial\Omega\}$ when $\theta \neq 1/2$. Now the conclusion is a consequence of 3) of Theorem 3.2: in particular $v(t, .) \in C^{2+\theta}(\bar{\Omega})$ by virtue of Schauder's theorem on elliptic equations.

*Remark.* If we choose as E a space of continuous functions in such that $\bar{D}_A = E$ (see e.g. [8]) then we must impose the further condition $g(t, x) = 0$ for $t \in [0, T]$ and $x \in \partial\Omega$. Hence the opportunity of introducing generators of analytic semigroups with non-dense domain.

Let us consider finally the abstract semilinear parabolic equation

$$(4.2) \qquad \begin{cases} u'(t) = Au(t) + \psi(u(t)) \quad , \quad 0 \le t \le T \\ u(0) = x \end{cases}$$

where A verifies (1.2) and $\psi: D_A(\theta) \to E$ is locally Lipschitz continuous. Problem (4.2) can be considered as the abstract version of the equation

$$(4.3) \qquad v_t(t, x) = \Delta v(t, x) + F(\nabla v(t, x)) \quad , \quad 0 \le t \le T, x \in \bar{\Omega}$$

with conditions $(4.1)_{2,3}$.

THEOREM 4.2. *For each* $x \in D_A$ *such that* $Ax + \psi(x) \in \bar{D}_A$ *there is* $T > 0$ *and a unique* $u \in C^1(0, T; E) \cap C(0, T; D_A)$ *which verifies* (4.2). *Moreover* $u', Au \in C^{1-\theta}(0^+, T; E)$. *If* $Ax + \psi(x) \in D_A(\theta, \infty)$, *then* $u', Au \in C^{1-\theta}(0, T; E)$.

*Proof.* Under our assumptions it was proved in [5] that there is $T > 0$ and a unique $u \in C(0, T; D_A(\theta))$ such that for $t \in [0, T]$:

$$u(t) = \exp(At)\,x + \int_0^t \exp(A(t-s))\,\psi(u(s))\,ds \tag{4.4}$$

Now $f = \psi \circ u \in C(0, T; E)$ hence from, 4) of Lemma 2.2 and (2.3) we deduce $u \in C^{1-\theta}(0, T; D_A(\theta))$ so that $f \in C^{1-\theta}(0, T; E)$. Now the conclusion follows from Theorem 3.2.

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