Enrico Magenes, Claudio Verdi, Augusto Visintin

Semigroup approach to the Stefan problem with non-linear flux


Accademia Nazionale dei Lincei

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Riassunto. — Un problema di Stefan a due fasi con condizione di flusso non lineare sulla parte fissa della frontiera è affrontato mediante la teoria dei semigruppi di contrazione in $L^1$. Si dimostra l’esistenza e l’unicità della soluzione nel senso di Crandall–Liggett e Bénilan.

Here we study the two-phase Stefan problem in more space variables with a non-linear flux condition on the fixed boundary. Denoting the space domain by $\Omega$ and the enthalpy density by $u$, we have a problem of the form

\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta \beta(u) = f & \text{in } \Omega \times ]0, T[ \\
\frac{\partial \beta(u)}{\partial n} + g(\beta(u)) = 0 & \text{on } \partial \Omega \times ]0, T[ \\
u(0) = u_0 & \text{in } \Omega ;
\end{cases}
\]

the non-decreasing function $\beta$ is characteristic of the material, $\beta(u)$ represents the temperature, $f$ is a datum and $g$ is a given (in general non-linear) function, as for the classical Stefan-Boltzmann radiation law.

Following the classical variational formulation in $L^2(\Omega)$ (for a discussion and further references see [12], e.g.), problem (P) has been recently studied in [5, 14, 15]. Here we use an approach based on the theory of non-linear contraction semigroups in $L^1(\Omega)$, following ideas and techniques used

(*) Pervenuta all’Accademia il 14 luglio 1983.  
(**) Dipartimento di Matematica dell’Università di Pavia e Istituto di Analisi Numerica del C.N.R. di Pavia.  
(***) Istituto di Matematica, Informatica e Sistemistica della Università di Udine.  
(****) Istituto di Analisi Numerica del C.N.R. di Pavia.
for similar problems in [2, 3, 4, 7, 8, 9]. We show that the operator
\[ A w = -\Delta \beta(w) \] with domain
\[
D(A) := \{ w \in L^1(\Omega) \mid \beta(w) \in W^{1,1}(\Omega), \Delta \beta(w) \in L^1(\Omega), \frac{\partial \beta(w)}{\partial v} + g(\beta(w)) = 0 \text{ on } \Gamma \}
\]
generates a contraction semigroup in \( L^1(\Omega) \); this yields the existence and uniqueness of the generalized solution of problem (P) in the sense of Crandall-Liggett and Bénilan. This approach seems especially useful for the numerical solution (see [3, 13]).

§ 1. THE CASE OF NO INTERNAL SOURCE \((f = 0)\)

Let \( \Omega \subset \mathbb{R}^N \) be a bounded regular domain for instance of class \( C^\infty \), with boundary \( \Gamma \). Let

\[
\begin{cases}
\beta : \mathbb{R} \rightarrow \mathbb{R} \text{ Lipschitz-continuous and non-decreasing, } \beta(0) = 0 \\
|\beta(\xi)| \geq C_1 |\xi| - C_2 \quad \forall \xi \in \mathbb{R} (C_1, C_2 \text{ positive constants})
\end{cases}
\]

(it is not restrictive to assume that the Lipschitz-constant of \( \beta \) is 1)

\[
\begin{cases}
g \in C^1(\mathbb{R}) \text{ non-decreasing, } g(0) = 0 \\
|g(\xi)| \leq C_3 |\xi| + C_4 \quad \forall \xi \in \mathbb{R} (C_3, C_4 \text{ positive constants})
\end{cases}
\]

(an explicit dependence of \( g \) on \( \sigma \in \Gamma \) would cause no further difficulty). We introduce the non-linear operator \( A : w \rightarrow \Delta \beta(w) \) with domain

\[
D(A) := \{ w \in L^1(\Omega) \mid \beta(w) \in W^{1,1}(\Omega), \Delta \beta(w) \in L^1(\Omega), \frac{\partial \beta(w)}{\partial v} + g(\beta(w)) = 0 \text{ on } \Gamma \}.
\]

Here the trace \( \beta(w) \) and the external normal trace \( \frac{\partial \beta(w)}{\partial v} \) are understood in the sense of Gagliardo (see [10] e.g.) and are in \( L^1(\Gamma) \); by the growth assumption on \( g \), also \( g(\beta(w)) \in L^1(\Gamma) \). The condition on \( \Gamma \) can also be written in the form

\[
\int_{\Omega} \nabla \beta(w) \cdot \nabla v \, dx + \int_{\Gamma} g(\beta(w)) \cdot v d\sigma = -\int_{\Omega} \Delta \beta(w) \cdot v dx \quad \forall v \in C^1(\bar{\Omega}).
\]
**Theorem 1.** $A$ is $m$-accretive in $L^1(\Omega)$, that is

\[
\forall f \in L^1(\Omega), \forall \lambda > 0, \exists ! w \in D(A) \text{ such that }
\]

\[
\begin{align*}
\forall \lambda > 0, (I + \lambda A)^{-1} \text{ is a contraction in } L^1(\Omega) \quad (I = \text{Identity}).
\end{align*}
\]

**Proof.** This is split into several steps.

(i) **Uniqueness of the solution of (4).**

Let $w_1, w_2$ be two solutions; setting $\theta_i = \beta(w_i)$ ($i = 1, 2$) we have

\[
\begin{align*}
\theta_i - \lambda \Delta \theta_i &= f - w_i + \theta_i = \Phi_i \quad \text{in } \Omega \\
\frac{\partial \theta_i}{\partial n} &= g(\theta_i) = \psi_i \quad \text{on } \Gamma.
\end{align*}
\]

Let $\{\Phi_{i,n} \in L^2(\Omega)\}_{n \in \mathbb{N}}, \{\psi_{i,n} \in H^1(\Gamma)\}_{n \in \mathbb{N}}$ be such that $\Phi_{i,n} \to \Phi_i$ strongly in $L^1(\Omega), \psi_{i,n} \to \psi_i$ strongly in $L^1(\Gamma)$; by well-known results (see [11], e.g.), the elliptic problem

\[
\begin{align*}
\theta_{i,n} - \lambda \Delta \theta_{i,n} &= \Phi_{i,n} \quad \text{in } \Omega \\
\frac{\partial \theta_{i,n}}{\partial n} &= \psi_{i,n} \quad \text{on } \Gamma
\end{align*}
\]

has one and only one solution $\theta_{i,n} \in H^2(\Omega)$. By Lemma 2.3 of [4] we have

\[
\| \theta_i - \theta_{i,n} \|_{W^{1,1}(\Omega)} \leq C (\| \Phi_{i,n} - \Phi_i \|_{L^1(\Omega)} + \| \psi_{i,n} - \psi_i \|_{L^1(\Gamma)}),
\]

with $C$ constant independent of $i, n$; therefore

\[
\theta_{i,n} \to \theta_i \quad \text{strongly in } W^{1,1}(\Omega) \quad \text{as } n \to \infty.
\]

We approximate the Heaviside graph $H$ as follows

\[
\begin{align*}
\{H_j \in C^1(\mathbb{R})\}_{j \in \mathbb{N}}, \quad H_j' \geq 0, \quad H_j(\xi) = 0 \quad \text{for } \xi \leq 0, \\
H_j(\xi) = 1 \quad \text{for } \xi \geq \frac{1}{j}.
\end{align*}
\]
Taking the difference between (8) written for $i=1,2$ and multiplying by $H_j(\theta_{1,n} - \theta_{2,n})$, we get

\begin{equation}
\int_{\Omega} (\theta_{i,n} - \theta_{2,n}) \cdot H_j(\theta_{i,n} - \theta_{2,n}) \, dx + \lambda \int_{\Gamma} (\psi_{i,n} - \psi_{2,n}) \cdot H_j(\theta_{i,n} - \theta_{2,n}) \, d\sigma = \int_{\Omega} (\Phi_{i,n} - \Phi_{2,n}) \cdot H_j(\theta_{i,n} - \theta_{2,n}) \, dx ;
\end{equation}

as $H_j \geq 0$, the second integral is non-negative; we can assume that the sequences $\{\Phi_{i,n}\}$ and $\{\psi_{i,n}\}$ are dominated by integrable functions for $i=1,2$; thus taking $n \to \infty$ in (13) we get

\begin{align*}
\int_{\Omega} (\theta_{i} - \theta_{2}) \cdot H_j(\theta_{i} - \theta_{2}) \, dx + \lambda \int_{\Gamma} [g(\theta_{i}) - g(\theta_{2})] \cdot H_j(\theta_{i} - \theta_{2}) \, d\sigma \\
\leq \int_{\Omega} (\Phi_{1} - \Phi_{2}) \cdot H_j(\theta_{1} - \theta_{2}) \, dx = \int_{\Omega} [(\theta_{1} - \theta_{2}) - (w_{1} - w_{2})] \cdot H_j(\theta_{1} - \theta_{2}) \, dx .
\end{align*}

The second integral is non-negative by the monotonicity of $g$ and the second member is non-positive by the properties of $\beta$; thus taking $j \to \infty$ we get

\[ \int_{\Omega} (\theta_{i} - \theta_{2})^{+} \, dx \leq 0 . \]

Interchanging $\theta_{1}$ and $\theta_{2}$ we have $\theta_{1} = \theta_{2}$ a.e. in $\Omega$, whence by (6) $w_{1} = w_{2}$ a.e. in $\Omega$.

(ii) $\forall f \in L^{2}(\Omega)$, $\forall \lambda > 0$, $\exists w \in D(A)$ solution of (4).

Using a standard procedure, we approach $\beta$ and $g$ by two sequences described by a positive parameter $\varepsilon$ as follows

\[ \beta_{\varepsilon} \in C^{\infty}(R), \ 0 < \varepsilon \leq \beta' \leq 1 , \quad \beta_{\varepsilon}(0) = 0 , \ \beta_{\varepsilon} \to \beta \ \text{uniformly in} \ R , \]

\[ g_{\varepsilon} \in C^{\infty}(R) , \ g_{\varepsilon}' \geq 0 , \ g_{\varepsilon}(0) = 0 , \ g_{\varepsilon} \to g \ \text{uniformly in} \ R ; \]

we also assume that $\beta_{\varepsilon}$ is uniformly Lipschitz-continuous and that $g_{\varepsilon}$ fulfills an order of growth assumption as in (2); moreover let

\[ f_{\varepsilon} \in C^{\infty}(R) , \ f_{\varepsilon} \to f \ \text{strongly in} \ L^{2}(\Omega) . \]
We consider the $\varepsilon$-regularized problems corresponding to (4); setting $\theta_\varepsilon \equiv \beta_\varepsilon(w_\varepsilon)$, $R_\varepsilon \equiv \beta_\varepsilon^{-1} - 1$, this can be written also in the form

\begin{align}
\theta_\varepsilon - \lambda \Delta \theta_\varepsilon + R_\varepsilon(\theta_\varepsilon) &= f_\varepsilon & \text{in } \Omega \\
\frac{\partial \theta_\varepsilon}{\partial n} + g_\varepsilon(\theta_\varepsilon) &= 0 & \text{on } \Gamma;
\end{align}

by well-known results (see [10], e.g.), this problem has one and only one solution $\theta_\varepsilon \in C^1(\bar{\Omega})$, for instance. Multiplying (15) by $\theta_\varepsilon$, by a standard procedure we get the a priori estimate

\begin{equation}
\| \theta_\varepsilon \|_{H^1(\Omega)} \leq C_\lambda \quad \text{(constant dependent on } \lambda \text{ but not on } \varepsilon),
\end{equation}

whence $\| \theta_\varepsilon \|_{L^2(\Gamma)} \leq C_\lambda$ and by the assumptions on $g_\varepsilon$

\begin{equation}
\| g_\varepsilon(\theta_\varepsilon) \|_{L^2(\Gamma)} \leq C_\lambda;
\end{equation}

by the assumptions on $\beta$ and $\beta_\varepsilon$, (17) entails also

\begin{equation}
\| w_\varepsilon \|_{L^2(\Omega)} \leq C_\lambda.
\end{equation}

By the previous a priori estimates there exist $w, \theta, \eta$ such that, possibly taking subsequences, as $\varepsilon \to 0$

\begin{align}
&w_\varepsilon \rightharpoonup w \quad \text{weakly in } L^2(\Omega) \\
&\theta_\varepsilon \equiv \beta_\varepsilon(w_\varepsilon) \rightharpoonup \theta \quad \text{weakly in } H^1(\Omega) \\
&g_\varepsilon(\beta_\varepsilon(w_\varepsilon)) \rightharpoonup \eta \quad \text{weakly in } L^2(\Gamma).
\end{align}

Using standard monotonicity techniques, one can show that

\begin{equation}
\theta = \beta(w) \quad \text{a.e. in } \Omega, \ \eta = g(\beta(w)) \quad \text{a.e. on } \Gamma,
\end{equation}

therefore taking $\varepsilon \to 0$ in (15), (16) a solution of (4) is obtained with the further regularity

\begin{align}
&w \in L^2(\Omega), \quad \beta(w) \in H^1(\Omega), \quad \Delta \beta(w) \in L^2(\Omega). \\
&(iii) \ \forall \lambda > 0, \ (I + \lambda A)^{-1}: \ L^2(\Omega) \to L^2(\Omega) \text{ is a contraction with respect to the norm of } L^1(\Omega),
\end{align}

i.e. for any $f_1, f_2 \in L^2(\Omega)$, denoting the corresponding solutions of (4) by $w_1, w_2$, we have

\begin{equation}
\|w_1 - w_2\|_{L^1(\Omega)} \leq \|f_1 - f_2\|_{L^1(\Omega)}.
\end{equation}
In order to prove this, we consider \( f_{1,\varepsilon}, f_{2,\varepsilon} \) as in (14) and denote the corresponding solutions of (15), (16) by \( w_{1,\varepsilon}, w_{2,\varepsilon} \). Taking the difference between (15) written for \( i = 1, 2 \) and multiplying by \( H_j(\theta_{1,\varepsilon} - \theta_{2,\varepsilon}) \), we get

\[
\int_{\Omega} (w_{1,\varepsilon} - w_{2,\varepsilon}) \cdot H_j(\theta_{1,\varepsilon} - \theta_{2,\varepsilon}) \, dx + \lambda \int_{\Omega} \nabla (\theta_{1,\varepsilon} - \theta_{2,\varepsilon}) \cdot \nabla H_j(\theta_{1,\varepsilon} - \theta_{2,\varepsilon}) \, dx + \\
+ \lambda \int_{\Gamma} [g_{\varepsilon}(\theta_{1,\varepsilon}) - g_{\varepsilon}(\theta_{2,\varepsilon})] \cdot H_j(\theta_{1,\varepsilon} - \theta_{2,\varepsilon}) \, d\sigma = \\
= \int_{\Omega} (f_{1,\varepsilon} - f_{2,\varepsilon}) \cdot H_j(\theta_{1,\varepsilon} - \theta_{2,\varepsilon}) \, dx,
\]

whence, as the second and third integrals are non-negative,

\[
(25) \quad \int_{\Omega} (w_{1,\varepsilon} - w_{2,\varepsilon}) \cdot H_j(\theta_{1,\varepsilon} - \theta_{2,\varepsilon}) \, dx \leq \int_{\Omega} (f_{1,\varepsilon} - f_{2,\varepsilon}) \cdot H_j(\theta_{1,\varepsilon} - \theta_{2,\varepsilon}) \, dx \leq \\
\leq \| f_{1,\varepsilon} - f_{2,\varepsilon} \|_{L^1(\Omega)}.
\]

Note that, denoting the Heaviside graph by \( H \), there exists \( \chi \in H(\theta_{1,\varepsilon} - \theta_{2,\varepsilon}) \) such that

\[
H_j(\theta_{1,\varepsilon} - \theta_{2,\varepsilon}) \rightarrow \chi \quad \text{weakly star in } L^\infty(\Omega);
\]

by the strict monotonicity of \( \beta_{\varepsilon} \) we have also \( \chi \in H(w_{1,\varepsilon} - w_{2,\varepsilon}) \), hence taking \( j \rightarrow \infty \) in (25) we get

\[
(26) \quad \int_{\Omega} (w_{1,\varepsilon} - w_{2,\varepsilon})^+ \, dx \leq \| f_{1,\varepsilon} - f_{2,\varepsilon} \|_{L^1(\Omega)}
\]

Interchanging \( w_{1,\varepsilon} \) and \( w_{2,\varepsilon} \) we have

\[
(27) \quad \int_{\Omega} (w_{2,\varepsilon} - w_{1,\varepsilon})^+ \, dx \leq \| f_{1,\varepsilon} - f_{2,\varepsilon} \|_{L^1(\Omega)}
\]

and then taking \( \varepsilon \rightarrow 0 \) in (26), (27) we get (24).

(iv) \( \forall f \in L^1(\Omega), \forall \lambda > 0, \exists w \in D(\Lambda) \) such that \( w - \lambda \Delta \beta(w) = f \) a.e. in \( \Omega \).

Let \( \{ f_n \in L^2(\Omega) \}_{n \in \mathbb{N}}, f_n \rightarrow f \) strongly in \( L^1(\Omega) \); denote by \( w_n \) the solution of (4) corresponding to \( f_n \). Thus, setting \( \theta_n = \beta(w_n), \theta_n \in \{ \theta \in W^{1,1}(\Omega) | \Delta \theta \in L^1(\Omega), \frac{\partial \theta}{\partial \nu} + g(\theta) = 0 \text{ on } \Gamma \} \) and

\[
\theta_n - \lambda \Delta \theta_n = f_n - w_n + \theta_n \quad \text{in } \Omega.
\]
By (iii) \( \{w_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( L^1(\Omega) \), thus there exists \( w \in L^1(\Omega) \) such that
\[
  w_n \to w \quad \text{strongly in} \ L^1(\Omega),
\]
whence, as \( \beta \) is Lipschitz-continuous, also
\[
  \theta_n = \beta(w_n) \to \theta = \beta(w) \quad \text{strongly in} \ L^1(\Omega);
\]
therefore
\[
  f_n - w_n + \theta_n \to f - w + \theta \quad \text{strongly in} \ L^1(\Omega)
\]
and taking \( n \to \infty \) in (28) we get that \( w \) solves (4) since \( -\Delta \) is \( m \)-accretive in \( L^1(\Omega) \) with domain \( D \) (see [4], e.g).

(v) \( \forall \lambda > 0 \), \( (I + \lambda A)^{-1} \) is a contraction in \( L^1(\Omega) \), i.e. \( \forall f_1, f_2 \in L^1(\Omega) \), denoting the corresponding solutions of (4) by \( w_1, w_2 \),
\begin{equation}
  \| w_1 - w_2 \|_{L^1(\Omega)} \leq \| f_1 - f_2 \|_{L^1(\Omega)}.
\end{equation}

In order to prove this, let \( \{f_{i,n} \in L^2(\Omega)\}_{n \in \mathbb{N}} \), \( f_{i,n} \to f_i \) strongly in \( L^1(\Omega) \) \( (i = 1, 2) \); let \( w_{i,n} \) denote the solution of (4) corresponding to \( f_{i,n} \). As we proved in (iv)
\[
  w_{i,n} \to w_i \quad \text{strongly in} \ L^1(\Omega);
\]
by (iii)
\[
  \| w_{1,n} - w_{2,n} \|_{L^1(\Omega)} \leq \| f_{1,n} - f_{2,n} \|_{L^1(\Omega)}
\]
and taking \( n \to \infty \) we get (29).

**Theorem 2.** \( D(A) \) is dense in \( L^1(\Omega) \).

**Proof.** As
\[
  D(A)_2 = \{ w \in L^1(\Omega) \mid \beta(w) \in H^1(\Omega), \Delta \beta(w) \in L^2(\Omega), \frac{\partial \beta(w)}{\partial n} + g'(\beta(w)) = 0 \quad \text{on} \Gamma \} \subset D(A)
\]
and the inclusion \( D(\Omega) \subset L^1(\Omega) \) is dense, it is sufficient to prove that
\[
  \forall f \in D(\Omega), \quad \text{setting} \ w_\lambda = (1 + \lambda A)^{-1} f \quad \text{with} \ w \in D(A)_2,
\]
then

\( w_\lambda \rightarrow f \) strongly in \( L^2(\Omega) \) as \( \lambda \rightarrow 0 \),

or equivalently

(30) \( \lambda \Delta \beta_x (w_\lambda) \rightarrow 0 \) strongly in \( L^2(\Omega) \).

To this aim we consider the regularized problems in \( \beta_x , g_x , f_x = f \) with solutions \( w_{\lambda, x} \), and we multiply the corresponding equation (15) by \( -\Delta \beta_x (w_{\lambda, x}) \), getting

\[
\int_\Omega \nabla w_{\lambda, x} \cdot \nabla \beta_x (w_{\lambda, x}) \, dx + \int_\Gamma g_x (\beta_x (w_{\lambda, x})) \cdot w_{\lambda, x} \, d\sigma + \lambda \int_\Omega [\Delta \beta_x (w_{\lambda, x})]^2 \, dx = \int_\Omega \nabla f \cdot \nabla \beta_x (w_{\lambda, x}) \, dx \leq \| \nabla f \|_{L^2(\Omega)} \cdot \| \nabla \beta_x (w_{\lambda, x}) \|_{L^2(\Omega)}.
\]

As \( g_x (0) = \beta_x (0) = 0 \) and \( g_x , \beta_x \) are monotone, the second integral is non-negative; moreover, by the properties of \( \beta_x \),

\[
\int_\Omega \nabla w_{\lambda, x} \cdot \nabla \beta_x (w_{\lambda, x}) \, dx \geq \int_\Omega |\nabla \beta_x (w_{\lambda, x})|^2 \, dx ;
\]

hence

\[
\| \nabla \beta_x (w_{\lambda, x}) \|_{L^2(\Omega)}^2 + \lambda \| \Delta \beta_x (w_{\lambda, x}) \|_{L^2(\Omega)}^2 \leq \| \nabla f \|_{L^2(\Omega)} \cdot \| \nabla \beta_x (w_{\lambda, x}) \|_{L^2(\Omega)}
\]

whence

\[
\| \nabla \beta_x (w_{\lambda, x}) \|_{L^2(\Omega)} \leq C \quad \text{(constant independent of } \lambda \text{ and } \varepsilon \text{)}
\]

and then also

\[
\lambda \| \Delta \beta_x (w_{\lambda, x}) \|_{L^2(\Omega)}^2 \leq C,
\]

which yields (30).

**Conclusion**

The operator \( A: D(A) \rightarrow L^1(\Omega) \) generates a non-linear semigroup of contractions \( S(t) \), defined by Crandall-Liggett's formula (see [6]):

\[
\forall u_0 \in L^1(\Omega) , \ S(t) u_0 = \lim_{n \to \infty} \left( I + \frac{t}{n} A \right)^{-n} u_0 \quad \text{uniformly in } [0, T].
\]
Moreover, \( u(t) = S(t)u_0 \in C^0([0,T]; L^1(\Omega)) \) is the generalized solution in the sense of Crandall-Liggett [6] and Bénilan [1] of the abstract Cauchy problem

\[
\frac{du}{dt} + Au = 0 \quad , \quad u(0) = u_0 ,
\]

or equivalently of problem (P) (see introduction) with \( f = 0 \).

\[\text{§ 2. The general case (} f \neq 0 \text{)}\]

Let \( f \in L^1(\Omega \times]0,T[) \); let \( f_n = f_{nk} \) constant in \( \left[k \frac{t}{n} , (k+1) \frac{t}{n}\right] \) for \( k = 0, \ldots , n-1 \) and such that \( f_n \to f \) strongly in \( L^1(\Omega \times]0,T[) \). Then

\[
\forall u_0 \in L^1(\Omega) \quad , \quad U_f(t)u_0 = \lim_{n \to \infty} \prod_{k=1}^{n} \left( I + \frac{t}{n} (A - f_{nk}) \right)^{-1} u_0
\]

(uniformly in \([0,T])\) is the generalized solution (see [7]) of the abstract Cauchy problem

\[
\frac{du}{dt} + Au = f \quad , \quad u(0) = u_0 ,
\]

i.e. of problem (P).

Remark. Under natural assumptions on \( u_0 \) and \( g \), the solution \( u \) of problem (P) with \( f = 0 \) fulfills a maximum principle: \( M_1 \leq u \leq M_2 (M_1, M_2: \text{constants}) \) (by means of an argument similar to one used in [15]). This allows the removal of the assumption on the growth of \( g \) (see (2)); therefore the above results apply also to the case of a flux governed by the classical Stefan-Boltzmann radiation law

\[
g(\tau) = C (\tau^4 - \tau_0^4) ;
\]

here \( \tau \) denotes the absolute temperature, \( \tau_0 \) is the temperature of a source and \( C > 0 \) is a physical constant.

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References