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Weakly hyperbolic equations of second order well-posed in some Gevrey classes

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<http://www.bdim.eu/item?id=RLINA_1983_8_75_1-2_19_0>
Equazioni a derivate parziali. — We&nbs...
moreover

$$\sup_{|t| = 1} \left\| \sum_{j=1}^{n} a_{ij}(x, t) \xi_{j} \xi_{j} \right\|_{BV([0, T] ; L^{\infty}(K))} = M K < + \infty ;$$

iii) For any $K$ compact subset of $\mathbb{R}^{n}$ there exist some positive constants $A_{K}$, $A_{K}$ such that

$$| D_{x}^{s} a_{ij}(x, t) | \leq \Lambda_{K} A_{K}^{s} | x |^{s} \forall (x, t) \in K \times [0, T], \forall x \in \mathbb{N}^{n}$$

for a fixed number $s \geq 1$ (i.e. the matrix $a_{ij}$ belongs to the Gevrey class $\gamma_{loc}^{(s)}$ in $x$, uniformly with respect to $t$).

From these hypotheses we have obtained the following

**Theorem 1.** Let $\varphi(x)$, $\psi(x) \in \gamma_{loc}^{(s)}$. Then problem (1) has one and only one solution $u \in C^{1}([0, T]; \gamma_{loc}^{(s)})$ provided that

$$1 \leq s < 1 + \frac{\sigma}{2}$$

**Remark 1.** If $a_{ij}(x, t) = a_{ij}(t)$, the result of Theorem 1 is contained in [2], where a class of counter-examples shows that this result, in a certain sense, cannot be improved.

In connection with Theorem 1, for coefficients holder continuous in $t$, see also [4].

2. **Sketch of the Proof**

The idea of the proof is to approximate problem (1) by means of a family of strictly hyperbolic problems with sufficiently smooth coefficients and to show that the corresponding solutions are bounded in $C^{1}([0, T]; \gamma_{loc}^{(s)})$, in order to obtain a sequence converging in $C^{1}([0, T]; \gamma_{loc}^{(s)})$ to a solution $u(x, t)$ of (1). After this, the uniqueness of the solution of (1) is obtained by a duality method.

In order to illustrate the situation, let us consider the simplest case of (1), i.e. the equation

\begin{align*}
\begin{cases}
    u_{tt} = a(t) u_{xx} & \text{on } \mathbb{R}_{x} \times [0, T] \\
    u(x, 0) = \varphi(x) \\
    u_{t}(x, 0) = \psi(x)
\end{cases}
\end{align*}
where \( a(t) \geq 0 \) and \( a(t)^{1/\alpha} \in \text{BV}([0, T]) \); moreover, as a further simplification, we suppose \( a(t) \in C^1([0, T]) \) (this last hypothesis is removable).

From [3] it is known that problem (1) is well-posed in \( \gamma^{(i)}_{loc} \) (the space of the real analytic functions) without any assumption of regularity in \( t \) as regards the coefficients \( a_{ij} \); therefore, we can suppose that \( s > 1 \) and that, according to the finite speed of propagation, \( \varphi(x) \) and \( \psi(x) \) have compact support, i.e. they belong to \( \gamma^{(i)}_0 \).

Equations of type (2) are studied in [2] by means of the Fourier-Laplace transform; now, let us see how we can obtain the result of Theorem 1 for the equation (2) using our method of approximation by strictly hyperbolic equations.

Let \( h \in \mathbb{N} \). Define:

\[
\begin{align*}
U_m(t) & = a(t) + h^{-\sigma}; \\
E_{h,m}(t) & = a_h(t) \int_{\mathbb{R}} [D_x^k u_m(x, t)]^2 dx + \int_{\mathbb{R}} [D_x^{k-1} D_t u_m(x, t)]^2 dx
\end{align*}
\]

(\( E_{h,m}(t) \) is a sort of approximated energy of \( D_x^{k-1} u_m \)).

From the fact that \( D_x^k u_m \) is a solution of (3) \( \forall k \in \mathbb{N} \) it follows easily that

\[
E_{h,m}(t) \leq \frac{|a_h(t)|}{a_h(t)} E_{h,m}(t) + h^{-\sigma/2} [E_{h,m}(t) + E_{h+1,m}(t)]
\]

if \( h \leq m - 1 \), while

\[
E_{m,m}(t) \leq \frac{|a'_m(t)|}{a_m(t)} E_{m,m}(t).
\]

Using Gronwall's lemma and the inequalities (4) (iterated \( m - 1 \) times) and (5) we obtain

\[
E_{1,m}(t) \leq \sum_{l=1}^{m} \lambda_l(t) E_{h,m}(0)
\]

where

\[
\lambda_l(t) = \exp \left\{ \int_0^t \frac{|a'_h(s)|}{a_h(s)} \, ds \right\} \cdot \bar{t}^{\bar{h} - 1} e \left[ (\bar{h} - 1)! \right]^{-1} \bar{t}^{-\sigma/2}.
\]
From the fact that \( a(t)^{1/\sigma} \in BV([0,T]) \) it follows
\[
\lambda_h(t) = e^{Bt} (te^{Bt})^{h-1} \cdot [(h - 1)!]^{(1+\sigma/2)}
\]
where \( B \) is a positive constant depending only on \( \| a(t)^{1/\sigma} \|_{BV([0,T])} \) while, being \( \varphi(x), \psi(x) \in \chi_0^{(\delta)} \), we can estimate
\[
E_{h,m}(0) \leq CA^{h-1} [(h - 1)!].
\]
Substituting (7) and (8) in (6) we get
\[
E_{1,m}(t) = Ce^{Bt} \sum_{m=1}^{\infty} \frac{(Ate^{Bt})^h}{(h!)^{1+\sigma/2-s}} \leq Ce^{Bt} \sum_{m=1}^{\infty} \frac{(Ate^{Bt})^h}{(h!)^{1+\sigma/2-s}}
\]
and the last series converges to a number independent of \( m \), this convergence being guaranteed by the fact that \( s < 1 + \sigma/2 \).
Analogously, one can prove other estimates on the \( E_{h,m}(t) \), independent of \( m \), of the type
\[
E_{h,m}(t) \leq C (2A)^h e^{Bht} (h!)^{1+\sigma/2-s} \sum_{m=1}^{\infty} (2Ate^{Bt})^h
\]
By means of (9) and (10) we get that the sequence \( u_m(x,t) \) is bounded in \( C^1([0,T]; \chi_0^{(\delta)}) \); therefore, there exists a subsequence \( u_m \) that converges to a function \( u(x,t) \in C^1([0,T]; \chi_0^{(\delta)}) \) which is a solution of (2).

This method of approximated energies in \( L^2 \)-norm of the solution and its derivatives, unlike the Fourier-Laplace transform, also works very well in the case in which the coefficients \( a_{ij} \) depend on \( x \); clearly, computations are much more complicated by the fact that the successive derivatives \( D_x^\alpha u \) of the solution \( u \) don't solve the original equation (1), but a modified equation with the same principal part of (1) plus other terms depending on the derivatives of the coefficients and of the solution up to the order \( |\alpha| = 1 \).

3. A Theorem for Strictly Hyperbolic Systems

Using just the same techniques, we are also able to prove the following theorem, regarding strictly hyperbolic systems:

**Theorem 2.** Let us consider the system
\[
\begin{align*}
U_t &= \sum_{1}^{n} A_h(x,t) U_{xh} + B(x,t) U \\
U(x,0) &= \varphi(x)
\end{align*}
\]
where \( A_h, B \) are \( N \times N \) matrices and \( \varphi \) is an \( N \)-vector.
We suppose that:

i) problem (11) is strictly hyperbolic, i.e. the equation

$$\det (\lambda I - \sum_{i=1}^{n} A_{\lambda}(x, t) \xi_{\lambda}) = 0$$

has $N$ real and distinct roots $\lambda = \lambda(x, t; \xi)$;

ii) $A_{\lambda}(x, t) \in C^{0,\alpha}([0, T]; \gamma_{loc}^{(\lambda)})$

$$B(x, t) \in L^1([0, T]; \gamma_{loc}^{(\lambda)})$$

(roughly speaking, the matrices $A_{\lambda}$ are hölder-continuous in $t$ and Gevrey in $x$).

Then, for any vector $\varphi(x) \in \gamma_{loc}^{(\lambda)}$ problem (11) has one and only one solution $U \in C([0, T]; \gamma_{loc}^{(\lambda)})$ provided that

$$1 \leq s < \frac{1}{1 - \alpha}.$$ 

This result, for a scalar operator of order 2, has been proved by T. Nishitani in [4], using quite different techniques; however, the first result in this direction, regarding second order hyperbolic equations with time dependent coefficients, is due to F. Colombini, E. De Giorgi and S. Spagnolo (see [1]).

REFERENCES


