Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Algebras of continuous functions over P-spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **74** (1983), n.6, p. 313–321.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1983_8_74_6_313_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1983.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 23 giugno 1983

Presiede il Presidente della Classe GIUSEPPE MONTALENTI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Algebras of continuous functions over P-spaces.* Nota di NICOLA RODINÒ, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Nella prima parte della nota sono studiate le proprietà di connessione dei sottospazi dello spettro di un anello. Con l'ausilio dei risultati ottenuti, si stabiliscono le condizioni necessarie e sufficienti affinchè un'algebra reale assolutamente piatta sia isomorfa ad un'algebra di funzioni continue a valori reali su un P–spazio, del quale determini la topologia. Ulteriori condizioni sono necessarie e sufficienti affinché un'algebra reale assolutamente piatta sia isomorfa all'algebra di tutte le funzioni continue su un P–spazio.

1. The question of characterizing, in an algebrical way, real algebras, which are $\mathscr{C}$ (X) for some completely regular space X, is unresolved (see survey article by A. W. Hager [1]). Here, we are concerned with the same question in the restricted class of absolutely flat algebras, obtaining the following result: an absolutely flat algebra is $\mathscr{C}$ (X), for some P-space X, if a) it has not a 'real radical' and satisfies b) a condition on ideals of a prescribed type and c) a completeness condition.

In the first part of this work, as a support for the second part, given a ring A, connectedness properties of subspaces of Spec(A) are studied.

2. Throughout this work, all rings are commutative and with an identity. Let A be a ring. As usual, we denote the set of prime ideals of A by Spec (A). Let X be contained in Spec (A). Consequently, we consider X endowed with

(*) Nella seduta del 12 marzo 1983.

the Zariski relative topology. Let a be an ideal of A. A typical closed (resp. open) set of the Zariski topology is $V_X(a) = \{p : p \in \operatorname{Spec}(A)\ \&\ a \subset p\}$ (resp. $D_X(a) = \{p : p \in \operatorname{Spec}(A)\ \&\ a \not\subset p\}$). We set ${}_X\sqrt{a} = \bigcap_{a \subset p, p \in X} p$. It is clear that the map from the set of ideals of A into the set of closed subsets of X, defined by $a \to V_X(a)$ is onto. Furthermore, when restricted to the set of ideals a such that $a = {}_X\sqrt{a}$ this map is one-one. Note that, if Y is a subset of X, the closure $\overline{Y}$ of Y in X is $V_X(\bigcap Y)$. We recall that, given two ideals a and b of A, $(b : a)$ denotes the ideal of all $x \in A$ such that xa is contained in b.

Let a be an ideal of A and X a subset of Spec(A).

2.1. LEMMA. $\bigcap D_X(a) = (\bigcap X : a)$.

Proof. Let $x \in (\bigcap X : a)$ and $p \in D_X(a)$. Take $\mathrm{a} \in a$ so that a does not belong to p. Since xa belongs to p, x must belong to p, p being a prime ideal. We deduce that $(\bigcap X : a) \subset \bigcap D_X(a)$.

Let $x \in \bigcap D_X(a)$ and $p \in X$. The ideal xa is contained in p, since, if $a \not\subset p$, x belongs to p. So $\bigcap D_X(a) \subset (\bigcap X : a)$.

Let $X \subset \operatorname{Spec}(A)$ and let a be an ideal of A.

2.2. DEFINITION. *a is called X-large iff, for each p belonging to X, a is not contained in p.*

In our notation a is X-large if $V_X(a) = \varnothing$.

2.3. PROPOSITION. a) *An open subset $D_X(a)$ of X is closed iff $a + (\bigcap X : a)$ is X-large.*

b) *A closed subset $V_X(a)$ of X is open iff $a + (\bigcap X : a)$ is X-large.*

Proof. a) If $D_X(a)$ is closed, then $D_X(a)$ is equal to its closure $\overline{D_X(a)} = V_X(\bigcap D_X(a))$ which, according to 2.1, is $V_X((\bigcap X : a))$. So $a + (\bigcap X : a)$ is X-large, because, if $(\bigcap X : a)$ is contained in $p \in X$, then a is not contained in p.

Let us suppose now that $a + (\bigcap X : a)$ is X-large. If $p \in V_X(\bigcap X : a)$, then $p \in D_X(a)$. This proves that $D_X(a)$ contains its closure $V_X(\bigcap X : a)$ and therefore is closed.

b) Apply a) to the open set $D_X(a)$ which is the complement of $V_X(a)$ in X and consequently is closed.

2.4. COROLLARY. *A closed point p of X is isolated iff $p + (\bigcap X : p)$ is X-large.*

2.5. COROLLARY. $\mathrm{a} \in A$ *is such that* $V_X(\mathrm{a})$ *is open in* X *iff* $A\mathrm{a} + (\bigcap X : \mathrm{a})$ *is X-large.*

Let A be a ring, a and b ideals of A, X a subspace of Spec(A).

2.6. LEMMA $V_X(a) = V_X(b)$ *implies* $(\bigcap X : a) = (\bigcap X : b)$.

$(\bigcap X : a) = (\bigcap X : b)$ *implies* $\overline{D_X(a)} = \overline{D_X(b)}$.

Proof. $V_X(a) = V_X(b) \Rightarrow D_X(a) = D_X(b) \Rightarrow \bigcap D_X(a) = \bigcap D_X(b)$. From 2.1, we conclude that $(\bigcap X : a) = (\bigcap X : b)$.

$$\overline{D_X(a)} = V_X(\bigcap D_X(a)) = V_X((\bigcap X : a)) = \overline{D_X(b)}.$$

Let us suppose also that $\bigcap X = \{0\}$ and let $V_X(a)$ be a clopen subset of X.

2.7. PROPOSITION. *There is an idempotent* e *of* A *such that* $V_X(a) = V_X(e)$ *iff* Ann (a) *is generated by an idempotent.*

Proof. $\Rightarrow$) From 2.6, Ann (a) is equal to Ann (e) which is generated by the idempotent $1 - e$.

$\Leftarrow$) Supposing Ann $(a) = Af$, with f idempotent, it is Ann $(a) =$ Ann $(1-f)$. Fom 2.6, $\overline{D_X(a)} = \overline{D_X(1-f)}$. By hypothesis $V_X(a)$ is clopen and, from b) of 2.3, $V_X(1-f)$ is also clopen $(Af + A(1-f) = A)$. Consequently $D_X(a)$ and $D_X(1-f)$ are closed. We conclude that $D_X(a) = D_X(1-f)$ or equally $V_X(a) = V_X(1-f)$.

3. Throughout this paragraph all algebras are commutative over the real field **R** and they have an identity 1_A. Algebra morphisms are **R**-linear maps, which are also ring morphisms preserving the identities. Let A be an algebra. A character of A is an algebra morphism from A into **R**. We denote by $\chi(A)$ the set of all characters of A. There is a natural map $\Phi : \chi(A) \rightarrow \text{Spec}(A)$ defined by $\Phi(\varkappa) = \text{Ker}\,\varkappa$. The coarsest topology on $\chi(A)$ for which the map Φ is continuous, is called the Zariski topology of $\chi(A)$. Since the map Φ is one-one, all results, stated about subspaces of Spec (A) in 2 §, have a natural translation in results about subspaces of $\chi(A)$. Also definitions have a natural translation: when we say that an ideal a is X-large, we mean that, for each $\varkappa \in X$, a is not contained in Ker $\varkappa$; analogously, if a is an ideal of A, we put ${}_X\sqrt{a} = \bigcap_{a \subset \text{Ker}\,\varkappa, \varkappa \in X} \text{Ker}\,\varkappa$. Let $X \subset \chi(A)$. When it is not otherwise clearly indicated, we always consider X endowed with the Zariski relative topology and, for each ideal a of A, we use the following notation to denote the closed (resp. open) subsets of $X : V_X(a) = \{\varkappa \in X : a \subset \text{Ker}\,\varkappa\}$ (resp. $D_X(a) = \{\varkappa \in X : a \not\subset \text{Ker}\,\varkappa\}$). Let A be an algebra and $X \subset \chi(A)$. The Gelfand map $\mathscr{G}_X : A \rightarrow \mathbf{R}^X$ is the algebra morphism defined by the formula: $\mathscr{G}_X(a) \cdot \varkappa = \varkappa(a)$, [2, Ch. 1, § 1, n° 5]. When the context is not ambiguous, we use the simpler notation $\hat{a} = \mathscr{G}_X(a)$ and we call the map $\hat{a} : X \rightarrow \mathbf{R}$ the Gelfand transform of a on X.

$V_X(a)$ is just the zero-set of $\hat{a}$ on X and consequently the kernel of $\mathscr{G}_X$ is $\bigcap_{\varkappa \in X} \mathrm{Ker}\, \varkappa = \{a \in A : V_X(a) = X\}$. Let A be an algebra and let X be contained in $\chi(A)$.

3.1. PROPOSITION. *The Gelfand transforms of the elements of* A *on* X *are locally constant iff, for each* $a \in A$, *the ideal* $Aa + (\bigcap_{\varkappa \in X} \mathrm{Ker}\, \varkappa : a)$ *is* X-*large.*

Proof. $\Rightarrow$) Let $a \in A$. If $\hat{a}$ is locally constant, then the closed set $\hat{a}^{-1}(0) = V_X(Aa)$ is open, hence 2.5 implies that $Aa + (\bigcap_{\varkappa \in X} \mathrm{Ker}\, \varkappa : a)$ is X-large.

$\Leftarrow$) We have to show that, for each $r \in \mathbf{R}$, $\hat{a}^{-1}(r)$ is open. We have $\hat{a}^{-1}(r) = \mathscr{G}_X(a - r\, 1_A)^{-1}(0) = V_X(a - r\, 1_A)$. In fact, $\hat{a}(\varkappa) = r$ means that $\varkappa(a) = r$ or analogously that $0 = \varkappa(a) - r = \varkappa(a - r\, 1_A) = \mathscr{G}_X(a - r\, 1_A) \cdot \varkappa$. By hypothesis, the ideal $A(a - r\, 1_A) + (\bigcap_{\varkappa \in X} \mathrm{Ker}\, \varkappa : a - r\, 1_A)$ is X-large. According to 2.5, the closed set $V_X(a - r\, 1_A)$, which is equal to $\hat{a}^{-1}(r)$, is open.

Let A be an algebra and $X \subset \chi(A)$. The A-weak topology on X is the coarsest one for which every $\hat{a}$, $a \in A$, is a continuous function on X, [3, 3.3, p. 38]. Below, we summarize some properties in a lemma.

3.2. LEMMA. *Let* $X \subset \chi(A)$. *The following assertions about* X *are true*:

a) *The* A-*weak topology is Hausdorff.*

b) *The Zariski topology is coarser than the* A-*weak one.*

c) *The Zariski and the* A-*weak topologies coincide iff* $\hat{A}$ *separates points from* A-*weakly closed sets. Furthermore, when the two topologies coincide, they are completely regular.*

d) *Suppose* $\bigcap_{\varkappa \in X} \mathrm{Ker}\, \varkappa = \{0\}$. *If* $Aa + \mathrm{Ann}(a)$ *is* X-*large for each* $a \in A$, *then the Zariski and the* A-*weak topologies coincide and they have* $\{V_X(a) : a \in A\}$ *as a base of open sets.*

Proof. a) Let $\varkappa, \lambda \in X$. If $\varkappa \neq \lambda$, then $\mathrm{Ker}\, \varkappa \neq \mathrm{Ker}\, \lambda$. Take $a \in \mathrm{Ker}\, \varkappa$, $a \notin \mathrm{Ker}\, \lambda$. It is $\hat{a}(\varkappa) = 0 \neq \hat{a}(\lambda)$. Hence the set of functions $\hat{A}$ separates points in X and consequently, in the A-weak topology, X is Hausdorff.

b) Let $V_X(a)$ a Zariski closed subset of X. It is $V_X(\mathit{a}) = \bigcap_{a \in \mathit{a}} V_X(a) = \bigcap_{a \in \mathit{a}} \hat{a}^{-1}(0)$. Since each $\hat{a}^{-1}(0)$ is A-weakly closed, $V_X(\mathit{a})$ is also A-weakly closed.

c) $\Rightarrow$) Let $\varkappa \notin F$, F A-weakly closed. The assumption F Zariski closed implies that $F = V_X(\mathit{a})$, for some ideal a of A. Since $\varkappa \notin F$, there is $a \in \mathit{a}$

with $\varkappa(a) \neq 0$. The Gelfand transform of $a/\varkappa(a)$ on X is zero on F and takes the value 1 at $\varkappa$.

$\Leftarrow$) From b), it is enough to show that every A-weakly closed subset F is Zariski closed. By hypothesis, for each $x \in \complement_X F$, there is $a_x \in A$ with $\hat{a}_x(x) = 1$ and $\hat{a}_x \equiv 0$ on F. Clearly $F = \bigcap_{x \notin F} V_X(a_x)$. Thus F, as intersection of Zariski closed sets, in Zariski closed too. The last assertion of c) is trivial.

d) It is:

$$D_X(a) = \{\varkappa \in X : \varkappa(a) \neq 0\} = \{\varkappa \in X : \hat{a}(\varkappa) \neq 0\} = \bigcup_{r \neq 0} \{\varkappa \in X : \hat{a}(\varkappa) = r\} =$$

$$= \bigcup_{r \neq 0} \{\varkappa \in X : \mathscr{G}_X(a - r\,1_A) \cdot \varkappa = 0\} = \bigcup_{r \neq 0} V_X(a - r\,1_A).$$

It is well known that the family $(D_X(a))_{a \in A}$ is a Zariski base of open sets [4, Ch. 1, Ex. 17]. Since according to 2.5 $V_X(a)$ is open for each $a \in A$, $(V_X(a))_{a \in A}$ is also a Zariski base of open sets. Apparently the family $(\hat{a}^{-1}(]s, t[))_{a \in A, s, t \in R}$ is a sub-base for the A-weak topology of X. Observe that $\hat{a}^{-1}(]s, t[) = \bigcup_{s<r<t} V_X(a - r\,1_A)$. Therefore the two topologies are equal.

Let A be an algebra and $X \subset \mathbf{X}(A)$.

3.3. DEFINITION. *A is said to be regular on X if the Zariski and the A-weak topologies on X are coincident.*

3.4. DEFINITION. *A character $\varkappa$ of A is said to be good if for each countably generated ideal $a \subset \operatorname{Ker} \varkappa$, there is $a \in A$ with $a \subset \sqrt{Aa} \subset \operatorname{Ker} \varkappa$. $\mathbf{X}_g(A)$ is the set of good characters of A.*

Recall that a P-*space* is a completely regular space in which every G_δ is open [3, 4J, p. 62].

3.5. PROPOSITION. *Let A be an algebra regular on $X \subset \mathbf{X}(A)$, $\bigcap_{\varkappa \in X} \operatorname{Ker} \varkappa = 0$. X, endowed with the Zariski topology, is a P-space iff:*

a) *For each $a \in A$, $Aa + \operatorname{Ann}(a)$ is X-large.*

b) *Every character belonging to X is good.*

Proof. $\Rightarrow$) Since X is a P-space, every continuous function on X is locally constant [3, 4J, p. 63]. In particular, for each $a \in A$, $\hat{a}$ is locally constant on X. From 3.1, a) is proved. Let *a* be a countably generated ideal of A, contained in $\operatorname{Ker} \varkappa$, $\varkappa \in X$. Let the family $(a_n)_{n \in N}$ generate *a*. For each $n \in \mathbf{N}$, $V_X(a_n)$ is open. Since $V_X(a) = \bigcap_{n \in N} V_X(a_n)$, $V_X(a)$ is a G_δ and hence an open neighborhood of $\varkappa$. From 3.2, d), there is $a \in A$, so that $V_X(a)$ is a neighbourhood of $\varkappa$ contained in $V_X(a)$. Thus we have $\varkappa \in V_X(a) \subset V_X(a)$ and consequently $a \subset {}_X\sqrt{a} \subset {}_X\sqrt{Aa} \subset \operatorname{Ker} \varkappa$. This proves that $\varkappa$ is good.

$\Leftarrow$) Let X be a set of good characters of A and suppose that a) is valid. From 3.2, c), X is completely regular. To prove that X is a P-space, it is enough to show that the intersection of a countable family $(V_X(a_n))_{n \in N}$ of neighbourhoods of a point $\varkappa$ of X, is also a neighbourhood of X. In fact, from 3.2, d), the family $(V_X(a))_{a \in \operatorname{Ker} \varkappa}$ is a neighbourhood base at $\varkappa$. Let a be the ideal generated by the family $(a_n)_{n \in N}$. Since $\varkappa$ is good, there is $a \in \operatorname{Ker} \varkappa$ such that $a \subset {}_X\sqrt{Aa}$ or equally $V_X(a) \subset V_X(a)$. $V_X(a)$ is open and so the proposition is demonstrated.

Remember that an algebra is said to be absolutely flat (also Von Neumann regular) if every principal ideal is generated by an idempotent [4, Ch. 2, Ex. 27].

3.6. LEMMA. *Let* A *be an absolutely flat algebra and*

$$X \subset \chi(A)\ , \bigcap_{\varkappa \in X} \operatorname{Ker} \varkappa = \{0\}.$$

For each $a \in A$, *then*:

a) $Aa + \operatorname{Ann}(a)$ *is* X-*large.*

b) ${}_X\sqrt{Aa} = Aa$.

Proof. Let e be an idempotent of A so that $Aa = Ae$.

a) Since $\operatorname{Ann}(e) = A(1_A - e)$, $Aa + \operatorname{Ann}(a) = A$.

b) Since $Aa \subset {}_X\sqrt{Aa}$, we have to show only that ${}_X\sqrt{Aa} \subset Aa$. Let $x \in {}_X\sqrt{Aa}$. We claim that $x = xe$. For each $\varkappa \in \chi(A)$, $\varkappa(e)$ is an idempotent of **R** and consequently it must be 0 or 1. For each $\varkappa \in X$, $\varkappa(x - xe) = \varkappa(x)\varkappa(1_A - e) = 0$, because, if $\varkappa(e) = 0$, then $\varkappa(x) = 0$. Since by hypothesis $\bigcap_{\varkappa \in X} \operatorname{Ker} \varkappa = \{0\}$, we conclude that $x - xe = 0$.

3.7. LEMMA. *Let* A *be an absolutely flat algebra and* $X \subset \chi(A)$, $\bigcap_{\varkappa \in X} \operatorname{Ker} \varkappa = \{0\}$.

a) A *is regular on* X:

b) *A character* $\varkappa$ *of* A *is good if every countably generated ideal contained in* $\operatorname{Ker} \varkappa$ *is contained in a principal one contained in* $\operatorname{Ker} \varkappa$.

Proof. a) from 3.6, a) and 3.2, d). b) from 3.6, b).

Before going on, we want to point out some facts. Let X be a topological space and A an algebra of continuous functions on X, separating points in X. For each x in X, let δ_x be the character of A defined by $\delta_x(a) = a(x)$. The map $\delta : x \to \delta_x$ from X into $\chi(A)$ is one-one, for A separates points in X. Furthermore, when $\chi(A)$ is endowed with the A-weak topology, δ is continuous and is an embedding if A determines the topology of X. We denote by $\mathscr{C}(X)$ the algebra of continuous functions on X.

Let A be an absolutely flat algebra.

3.8. THEOREM. A *is isomorphic to a subalgebra* A′ *of* $\mathscr{C}$ (X), *for some* P-*space* X, *whose topology* A′ *determines, iff* $\bigcap_{\varkappa \in \chi_g(A)} \operatorname{Ker} \varkappa = \{0\}$.

Proof. ⇒) To simplify the notation, consider A as a subalgebra of $\mathscr{C}$ (X). Since A determines the topology of X, the map $\delta : X \to \chi(A)$ is an embedding and consequently $\delta(X)$ is a P-space. From 3.7, a), A is regular on $\delta(X)$. Apply 3.5 to deduce that every character δ_x is good. If $a \in \bigcap_{\varkappa \in \chi_g(A)} \operatorname{Ker} \varkappa$, then, for each $x \in X$, $\delta_x(a) = a(x) = 0$. Therefore, $a = 0$.

⇐) If $\bigcap_{\varkappa \in \chi_g(A)} \operatorname{Ker} \varkappa = 0$, then the Gelfand map $\mathscr{G}_{\chi_g(A)} : A \to \hat{A}$ is an isomorphism. So $\hat{A}$ is absolutely flat and from 3.7, a), $\hat{A}$ is regular on $\chi_g(A)$. Since a) and b) of 3.5 are trivially fulfilled, $\chi_g(A)$ is a P-space.

Remember that a realcompact space X is a completely regular topological space such that the map $\delta : X \to \chi(\mathscr{C}(X))$ is onto [3, Ch. 9, p. 114].

Let A be an absolutely flat algebra and $(e_\lambda)_{\lambda \in L}$ a family of mutually orthogonal idempotents of A, generating a $\chi(A)$-large ideal. Consider the linear topology on A, for which the set of ideal $\operatorname{Ann}((e_\lambda)_{\lambda \in F})$, where F is a finite subset of L, is a neighbourhood base at 0. A is said to satisfy the *completeness condition* if it is complete, whenever endowed with a topology of the type described above.

3.9. THEOREM. *Let* A *be an absolutely flat algebra.* A *is isomorphic to* $\mathscr{C}$ (X), *for some realcompact* P-*space* X *iff*:

a) $\chi(A) = \chi_g(A)$ *and* $\bigcap_{\varkappa \in \chi(A)} \operatorname{Ker} \varkappa = \{0\}$.

b) *For each ideal* a *of* A, $a + \operatorname{Ann}(a)$ *is* $\chi(A)$-*large implies that* $\operatorname{Ann}(a)$ *is principal.*

c) A *satisfies the completeness condition.*

Proof. ⇒) For simplicity, let us suppose that $A = \mathscr{C}(X)$.

a) From 3.8, a fortiori $\bigcap_{\varkappa \in \chi(A)} \operatorname{Ker} \varkappa = \{0\}$. From 3.7 a), A is regular on $\chi(A)$, i.e. the A-weak topology on $\chi(A)$ reduces to the Zariski topology. Consider $\chi(A)$ provided with the Zariski topology. Since X is completely regular, A determines the topology of X. By hypothesis, the map $\delta : X \to \chi(A)$ is onto. Consequently δ is an homeomorphism and $\chi(A)$ is a P-space. By 3.5, $\chi(A) = \chi_g(A)$.

b) If $a + \operatorname{Ann}(a)$ is $\chi(A)$-large, from 2.3, $D(a)$ is clopen (now and in the sequel, we omit subscripts in denoting closed or open subsets of $\chi(A)$). Let e be the characteristic function of $\delta^{-1}(D(a))$. Since $\delta^{-1}(D(a))$ is clopen,

e is a continuous function and so it belongs to A. It is $V(e) = V(a)$ because of the following equivalences: $\delta_x \in V(e) \Longleftrightarrow \delta_x(e) = 0 \Longleftrightarrow e(x) = 0 \Longleftrightarrow$ $\Longleftrightarrow x \notin \delta^{-1}(D(a)) \Longleftrightarrow \delta_x \notin D(a) \Longleftrightarrow \delta_x \in V(a)$. Apply proposition 2.7 to deduce b).

c) Let $(e_\lambda)_{\lambda \in L}$ be a family of two by two orthogonal idempotents of A, generating a $\chi(A)$-large ideal a. Endow A with the linear topology, which has the neighbourhood base at 0 consisting of the ideals $\mathrm{Ann}((e_\lambda)_{\lambda \in M})$ for each finite $M \subset L$. Since X is a P-space, for each $\lambda \in L$, the set $e_\lambda^{-1}(1)$ is open. Since the functions e_λ are two by two orthogonal, the sets $e_\lambda^{-1}(1)$ are two by two disjoint. Furthermore the family $(e_\lambda^{-1}(1))_{\lambda \in L}$ is an open cover of X. Indeed, if $x \in X$, the $\chi(A)$-large ideal a is not contained in $\mathrm{Ker}\ \delta_x$. This means that there is an index λ so that $\delta_x(e_\lambda) = e_\lambda(x) \neq 0$. Since $e_\lambda(x)$ is an idempotent of **R**, it must be 1 and so $x \in e_\lambda^{-1}(1)$. Let $\mathscr{F}$ be a Cauchy filter on A. For each $\lambda \in L$, there is $F_\lambda \in \mathscr{F}$ so that $F_\lambda - F_\lambda \subset \mathrm{Ann}(e_\lambda)$. Take $g_\lambda \in F_\lambda$. Let f be the continuous function that agrees with g_λ on $e_\lambda^{-1}(1)$. We claim that $\mathscr{F}$ converges to f. Let M be a finite subset of L. $F = \bigcap_{\lambda \in M} F_\lambda$ belongs to $\mathscr{F}$. For each $\lambda \in M$ and $g \in F$, $(f - g) e_\lambda = 0$. In fact, if $e_\lambda(x) \neq 0$, $x \in e_\lambda^{-1}(1)$ and $(f-g)(x) = g_\lambda(x) - g(x)$. The function $g_\lambda - g$ belongs to $F_\lambda - F_\lambda$, which is contaned in $\mathrm{Ann}(e_\lambda)$. Thus $(g_\lambda - g) e_\lambda = 0$ and so $g_\lambda(x) - g(x) = 0$. In this way we have shown that $f - F$ is contained in $\mathrm{Ann}((e_\lambda)_{\lambda \in M})$. For the arbitrariness of M, the filter $\mathscr{F}$ converges to f.

$\Leftarrow$) Let A be an absolutely flat algebra and suppose that *a*), *b*) and *c*) are valid. From a) the Gelfand morphism $\mathscr{G}_{\chi(A)}$ is one-one. Consequently A is isomorphic to its image $\hat{A}$. Endow $\chi(A)$ with the Zariski topology. Since by 3.7 a), A is regular on $\chi(A)$, the conditions of 3.5 are fulfilled and $\chi(A)$ is a P-space. We claim that $\hat{A} = \mathscr{C}(\chi(A))$. If $D(a)$ is a clopen subset of $\chi(A)$, from b) it is possible to apply the proposition 2.7 and find an idempotent $e \in A$ such that $D(a) = D(e)$. Clearly the Gelfand transform $\hat{e}$ of e is the characteristic function of $D(a)$. Thus we can assert that the idempotents of $\mathscr{C}(\chi(A))$ belong to $\hat{A}$. Let $f \in \mathscr{C}(\chi(A))$. Since $\chi(A)$ is a P-space, for each $r \in \mathrm{Im}(f)$, the set $f^{-1}(r)$ is clopen. For each $r \in \mathrm{Im}(f)$, let e_r be the idempotent of A so that $\hat{e}_r$ is the characteristic function of $f^{-1}(r)$. Since $\chi(A) = \bigcup_{r \in \mathrm{Im}(f)} f^{-1}(r)$ and since the sets $f^{-1}(r)$, $r \in \mathrm{Im}(f)$, are two by two disjoint, the idempotents e_r are two by two orthogonal and generate an $\chi(A)$-large ideal. Endow $\mathscr{C}(\chi(A))$ with the topology for which the ideals $\mathrm{Ann}((\hat{e}_r)_{r \in F})$, where F is finite and contained in $\mathrm{Im}(f)$, are a fundamental system of neighbourhoods of 0. By virtue of the completeness condition, $\hat{A}$, provided with the relative topology, is complete. It follows that $\hat{A}$ is closed in $\mathscr{C}(\chi(A))$. If we show that f is a cluster point of $\hat{A}$ in $\mathscr{C}(\chi(A))$, then f must belong to $\hat{A}$. Let $U = \mathrm{Ann}((e_r)_{r \in F})$, with $F \subset \mathrm{Im}(f)$ and finite, be a neighbourhood of 0.

We have to prove that there is $a \in A$ so that $\hat{a} - f$ belongs to U. We put $a = \sum_{r \in F} r e_r$ and claim that $\hat{a} - f$ belongs to U. We have to prove that, for each $s \in F$, $(\hat{a} - f)\,\hat{e}_s = 0$. By the orthogonality, $(\hat{a} - f)\,\hat{e}_s = s\hat{e}_s - f\hat{e}_s$. If $\hat{e}_s(x) \neq 0$, then $\hat{e}_s(x) = 1$ and $x \in f^{-1}(s)$. Hence $(\hat{a} - f)(x) = 0$. This proves that, for each $x \in \chi(A)$, $(\hat{a}(x) - f(x))\,\hat{e}_s(x)$ is zero. The theorem is thus demonstrated.

REFERENCES

[1] A. W. HAGER (1979) – *A class of function algebras (and compactifications and uniform spaces), Simposia Mathematica N° 17.*

[2] N. BOURBAKI – *Théories Spectrales, Fasc. XXXII, Hermann.*

[3] M. JERISON and L. GILLMAN (1960) – *Rings of Continuous Functions, Van Nostrad.*

[4] M. F. ATIYAH and I. G. MACDONALD (1969) – *Introduction to Commutative Algebra, Addison–Wesley.*