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Synthesis of stochastic optimal control for a convex optimization problem in Hilbert spaces

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Analisi matematica. — *Synthesis of stochastic optimal control for a convex optimization problem in Hilbert spaces.* Nota di GIANLUCA GORNI (*), presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si studia il problema della sintesi per un problema di controllo stocastico con equazione di stato lineare e funzione costo convessa.

1. NOTATIONS

We shall denote by H a separable Hilbert space, by $(\Omega, \mathscr{F}, \mathbf{P})$ a complete probability space on which a Brownian motion w is defined. We shall denote by $\mathscr{F}(s, t), 0 \leq s \leq t \leq T$, the σ-field generated by $\{w_r - w_s ; s \leq r \leq t\}$ and by the **P**-null sets of $\mathscr{F}$ (in order to make the filtration continuous).

The spaces of processes we will consider are $M_t^2(s, T; H), 0 \leq t \leq s \leq T$, which is the Hilbert space of the measurable and Bochner square-integrable processes $X : [s, T] \to L^2(\Omega, \mathscr{F}, \mathbf{P}; H)$ which are adapted to the filtration $\{\mathscr{F}(t, r) ; r \in [s, T]\}$. The projection of $M_0^2(t, T; H)$ onto $M_t^2(t, T; H)$ will be denoted by P^t and it is seen to be given by $P^t X(r) = E(X(r) \mid \mathscr{F}(t, r))$ for a.e. r.

For Hilbert—valued processes and Brownian motion see for instance Curtain–Pritchard [3].

We recall that, given a convex function $f : H \to]-\infty, +\infty]$, the subdifferential of f at $x \in H$ is the set

$$\partial f(x) = \{y \in H ; \forall z \in H \quad f(z) \geq f(x) + \langle z - x, y \rangle_H\},$$

and the convex conjugate function of f is defined by

$$f_*(x) = \sup \{\langle x, y \rangle_H - f(y) ; y \in H\}.$$

See for instance Barbu–Precupanu [1].

2. STATEMENT OF THE PROBLEM

Let H, K be separable Hilbert spaces, $B : K \to H$ a continuous linear operator, A the generator of a strongly continuous semi-group of bounded linear operators on H, denoted by $S(t), t \geq 0$.

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Consider the stochastic state equation

$$(1)\qquad \begin{aligned} &dy = (Ay + Bu)\,dt + dw \quad , \quad u \in M_0^2(t_0, T; K)\,; \\ &y(t_0) = x_0 \in L^2(\Omega, \mathcal{F}(0, t_0), \mathbf{P}; H)\,. \end{aligned}$$

Equation (1) has the unique mild solution y (which is a mean-square continuous process)

$$(2)\qquad y(u, t_0, x_0; s) = S(s - t_0)\,x_0 + \int_{t_0}^{s} S(s-r)\,Bu(r)\,dr + \int_{t_0}^{s} S(s-r)\,dw_r$$

(see Curtain–Pritchard [3], p. 143).

Let $V, \varphi : H \to [0, +\infty[$, $F : K \to [0, +\infty[$ be continuous convex functions. We assume that V and F are continuous Fréchet-derivable and that moreover the following relations hold:

$$(3)\qquad \begin{aligned} &\exists c \geq 0 \qquad \forall x \in H \qquad V(x) \leq c(|x|^2 + 1) \\ &\qquad\qquad\qquad\qquad\qquad\ \ \varphi(x) \leq c(|x|^2 + 1) \\ &\exists a > 0 \ , \ b \in \mathbf{R} \qquad \forall u \in K \quad F(u) \geq a|u|^2 + b\,. \end{aligned}$$

We consider the following cost functional:

$$(4)\qquad J(t_0, x_0, u) = E\left(\int_{t_0}^{T} (V(y(u, t_0, x_0; s)) + F(u(s)))\,ds + \varphi(y(u, t_0, x_0; T)) \right),$$

for $t_0 \in [0, T]$, $x_0 \in L^2(\Omega, \mathcal{F}(0, t_0), \mathbf{P}; H)$ and $u \in M_0^2(t_0, T; K)$.

We are concerned with the problem:

$$(5)\qquad \text{minimize} \ \ J(0, x_0, u) \qquad \text{over all} \ \ u \in M_0^2(0, T; K)\,.$$

Let us define the value function W:

$$(6)\qquad W(t_0, x) = \inf\{J(t_0, x, u)\,;\, u \in M_0^2(t_0, T; K)\}\,,$$

for $t_0 \in [0, T]$ and $x \in H$ (we shall identify an element x of H with the constant function $\Omega \to H$, $\omega \mapsto x$).

$W(t_0, \cdot)$ will be seen to be a continuous convex function.

In this paper the following necessary condition for optimality, called synthesis of optimal control, is announced:

THEOREM 1. *If* $u \in M_0^2(0, T; K)$ *is optimal for problem* (5), *then for a.e.* $t \in [0, T]$

$$u(t)(\omega) \in \partial F_*(-B^* \partial W(t, \cdot)(y(t)(\omega))) \qquad \mathbf{P}\text{-a.s.}; \tag{7}$$

B^* *is the adjoint operator of* B, F_* *the convex conjugate of* F *and* y *the state correspondent to the control* u *through formula* (2).

Formula (7), giving the synthesis for the control problem (5), generalizes the analogous result for the deterministic case (see Barbu-Precupanu [1], pp. 286–292). In the sequel we will denote by $\overline{W}(t, \cdot)$ the natural extension of $W(t, \cdot)$ to all $x \in L^2(\Omega, \mathscr{F}(0, t), \mathbf{P}; H)$.

$u \in M_0^2(t, T; K)$ will be said to be optimal in (t, x) iff $\overline{W}(t, x) = J(t, x, u)$.

3. PONTRJAGIN AND BELLMAN OPTIMALITY PRINCIPLES

The existence of a control $u \in M_0^2(t, T; K)$ optimal in (t, x) is a consequence of the following standard lemma:

LEMMA 2. $J(t, x, \cdot)$ *is a proper, convex and lower-semicontinuous function on* $M_0^2(t, T; K)$, *and* $J(t, x, u) \geq a \|u\|^2 + (T - t) b$.

Moreover, proceeding as in Barbu-Precupanu [1] or as in Bensoussan [2], we get the following Pontrjagin-type optimality principle:

PROPOSITION 3. $u \in M_0^2(0, T; K)$ *is optimal for problem* (5) *if and only if the following conditions are verified*:

$$J(0, x_0, u) < +\infty \tag{8}$$

$$F'(u(t)) = -B^* E(p(t) \mid \mathscr{F}(0, t)) \qquad \text{for a.e. } t, \tag{9}$$

where the process p is given by

$$p'(t) = -A^* p(t) - V'(y(u, 0, x_0; t)) \tag{10}$$

$$p(T) \in -\partial\varphi(y(u, 0, x_0, ; T)) \tag{11}$$

and A^* *is the adjoint operator of* A.

In particular, if u is optimal, then $F' \circ u$ is a mean-square continuous process.

The following propositions are also easy to prove:

PROPOSITION 4. *If* $u \in M_0^2(t, T; K)$ *is optimal in* (t, x) *and* $t \leq s \leq T$, *then the restriction of* u *to* $[s, T]$ *is optimal in* $(s, y(u, t, x; s))$.

PROPOSITION 5. *$\overline{W}(t, \cdot)$ is a continuous convex function, and if $0 \leq t \leq s \leq T$, $x \in L^2(\Omega, \mathscr{F}(0, t), \mathbf{P}; H)$ then*

$$\overline{W}(t, x) = \min \left\{ E\left(\int_t^s (V(y(u, t, x; r)) + F(u(r)))\, dr \right) + \right.$$
$$\left. + \overline{W}(s, y(u, t, x; s)) ; u \in M_0^2(t, s; K) \right\}, \tag{12}$$

and the minimum is attained by the restrictions to $[t, s]$ of the controls which are optimal in (t, x).

4. THE CONDITIONAL VALUE FUNCTION

Let $t \in [0, T]$, $x \in L^2(\Omega, \mathscr{F}(0, t), \mathbf{P}; H)$, $u \in M_0^2(t, T; K)$, we shall call sample cost function the random variable defined by

$$\mathscr{T}(t, x, u) = \int_t^T (V(y(u, t, x; s)) + F(u(s)))\, ds + \varphi(y(u, t, x; T)) \tag{13}$$

and conditional cost function the one defined by

$$\mathscr{J}(t, x, u) = E(\mathscr{T}(t, x, u) \mid \mathscr{F}(0, t)). \tag{14}$$

It is clear that $J(t, x, u) = E(\mathscr{J}(t, x, u))$. Moreover the following lemmas hold:

LEMMA 6. *If $C \in \mathscr{F}(0, t)$, $x, \overline{x} \in L^2(\Omega, \mathscr{F}(0, t), \mathbf{P}; H)$ are such that $x = \overline{x}$ on C and $u, \bar{u} \in M_0^2(t, T; K)$ are such that $u(s) = \bar{u}(s)$ on C for a.e. $s \in [t, T]$, then $\mathscr{J}(t, x, u) = \mathscr{J}(t, \overline{x}, \bar{u})$ on C.*

$\mathscr{J}(t, x, u)$ is an $\mathscr{F}(0, t)$-measurable real random variable, defined up to **P**-equivalence. The set L of the real $\mathscr{F}(0, t)$-measurable real random variables, identified up to **P**-equivalence, is naturally endowed with a partial order relation.

LEMMA 7. *There exists*

$$\mathscr{W}(t, x) = \min \{\mathscr{J}(t, x, u) ; u \in M_0^2(t, T; K)\} \tag{15}$$

in the sense of the vector lattice L. *Moreover, the function $\mathscr{W}$* (called conditional value function; see also Davis [4], p. 93) *verifies*:

$$E(\mathscr{W}(t, x)) = \overline{W}(t, x) \quad \text{and} \tag{16}$$

$$\mathscr{J}(t, x, u) = \mathscr{W}(t, x) \quad \text{iff} \quad J(t, x, u) = \overline{W}(t, x). \tag{17}$$

LEMMA 8. *If* C, x, $\bar{x}$ *are as in Lemma 6, then* $\mathscr{W}(t, x) = \mathscr{W}(t, \bar{x})$ *on* C.

LEMMA 9. *Let* $0 \leq t \leq T$, $x \in L^2(\Omega, \mathscr{F}(0, t), \mathbf{P}; H)$, $X \in \partial\overline{W}(t, \cdot)(x)$. Then for every $\bar{x} \in L^2(\Omega, \mathscr{F}(0, t), \mathbf{P}; H)$

$$\mathscr{W}(t, \bar{x}) - \mathscr{W}(t, x) \geq \langle \bar{x} - x, X \rangle_H \qquad \mathbf{P}\text{-a.s.} \tag{18}$$

Moreover, the mapping $\mathscr{W}(t, \cdot) : L^2(\Omega, \mathscr{F}(0, t), \mathbf{P}; H) \to L^1(\Omega, \mathscr{F}(0, t), \mathbf{P}; \mathbf{R})$ is continuous.

PROPOSITION 10. *If* $x \in H$, $u \in M_0^2(t, T; K)$ *then*

$$J(t, x, P^t u) \leq J(t, x, u) \tag{19}$$

P^t *being the projection defined in paragraph 1.*

Consequently, there exists $\bar{u} \in M_t^2(t, T; K)$ which is optimal in (t, x). Moreover $\mathscr{W}(t, x) = W(t, x)$ $\mathbf{P}$-a.s.

We get finally:

PROPOSITION 11. *Let* $0 \leq t \leq T$ *and* $x \in L^2(\Omega, \mathscr{F}(0, t), \mathbf{P}; H)$; *then*

$$\overline{W}(t, x) = \int_\Omega W(t, x(\omega))\, d\mathbf{P}(\omega) \qquad \text{and} \tag{20}$$

$$\mathscr{W}(t, x)(\omega) = W(t, x(\omega)) \qquad \mathbf{P}\text{-a.s.} \tag{21}$$

5. PROOF OF THEOREM 1

The result of proposition 11 enables us to rewrite proposition 5 in this way:

$$W(0, x) = \min \left\{ E\left(\int_0^s (V(y(u, 0, x; r)) + F(u(r)))\, dr + \right.\right. \left.\left. + W(s, y(u, 0, x; s)) \right); u \in M_0^2(0, s; K) \right\}, \tag{22}$$

that is, problem (5) with T replaced by s and φ by $W(s, \cdot)$.

We get the desired result combining formulas (9) and (11), keeping in mind that $F' \circ u$ is mean-square continuous.

References

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