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Non-subharmonicity of the Hausdorff distance


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**Geometria. — Non-subharmonicity of the Hausdorff distance.** Nota (*) del **Corrisp. EDOARDO VESENTINI.**

**RIASSUNTO. — Si dimostra con esempi che la distanza di Hausdorff–Carathéodory fra i valori di funzioni multivoche, analitiche secondo Oka, non è subharmonica.**

Let $D$ be a bounded domain of $\mathbb{C}$, and let $c$ be the Carathéodory distance on $D$. According to Theorem I of [5], the function $(x, y) \mapsto \log c(x, y)$ is plurisubharmonic on $D \times D$. Thus, for any domain $U$ of $\mathbb{C}$ and for all holomorphic maps $f, g$ of $U$ into $D$, the function $\zeta \mapsto \log c(f(\zeta), g(\zeta))$ is subharmonic on $U$.

Let $f, g$ be analytic set-valued functions on $U$, in the sense of K. Oka [2, 3], taking their values in the family of all compact subsets of $D$. Let $h_d$ and $h_c$ be the Hausdorff distances between subsets of $D$, defined respectively in terms of the euclidean distance $d$ on $\mathbb{C}$ and of the Carathéodory distance $c$ on $D$: for $H^j \subset D$ ($j = 1, 2$),

$$
\begin{align*}
  h_d(H^1, H^2) &= \max \{ \sup \{ d(z, H_2) : z \in H_1 \}, \sup \{ d(H_1, z) : z \in H_2 \} \}, \\
  h_c(H^1, H^2) &= \max \{ \sup \{ c(z, H_2) : z \in H_1 \}, \sup \{ c(H_1, z) : z \in H_2 \} \},
\end{align*}
$$

where

$$
\begin{align*}
  d(z, H^j) &= d(H^j, z) = \inf \{ |z - u| : u \in H^j \}, \\
  c(z, H^j) &= c(H^j, z) = \inf \{ c(z, u) : u \in H^j \}, \\
  (j = 1, 2).
\end{align*}
$$

The question arises whether the functions $\zeta \mapsto h_d(f(\zeta), g(\zeta))$, $\zeta \mapsto h_c(f(\zeta), g(\zeta))$, are subharmonic on $U$. The present paper will provide a negative answer to this question, contrary to a statement added in proof to [6]. The basic tool will be a result of [3], whereby for any complex Banach algebra $A$ and any holomorphic map $F : U \rightarrow A$ the function $f : \zeta \mapsto \text{Sp} F(\zeta)$, mapping $\zeta \in U$ onto the spectrum $\text{Sp} F(\zeta)$ of $F(\zeta)$, is Oka-analytic. Let $F$ be such that $\text{Sp} F(\zeta) \subset D$ for all $\zeta \in U$. Examples will be constructed showing that, for some compact set $K \subset D$, both the functions

$$
(1) \quad \zeta \mapsto h_d(\text{Sp} F(\zeta), K), \quad \zeta \mapsto h_c(\text{Sp} F(\zeta), K)
$$

are not subharmonic on $U$. This fact entails that Theorem I of [4] —which is a consequence of Theorem I of [5]—does not extend to Oka-analytic functions.

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1. For $r > 0$, let
\[ C(r, s) = \{ z \in \mathbb{C} : r < |z| < s \} \quad \text{and} \quad \Delta(r) = \{ z \in \mathbb{C} : |z| < r \} . \]

In nn. 1, 2, $U$ will be the unit disc $\Delta = \Delta(1)$. Let $\Lambda = \ell^\infty$, the commutative unital complex Banach algebra of all bounded sequences $x = \{x_n\}$ of complex numbers $x_n (n = 0, 1, \ldots)$ with component-wise addition and multiplication, and with norm $\|x\| = \sup |x_n|$. Let $a = \{a_n\}$ and $b = \{b_n\}$ be dense, respectively, in $\Delta \left( \frac{1}{3} \right)$ and $C \left( \frac{1}{4}, \frac{1}{2} \right)$. Then $Sp b = C \left( \frac{1}{4}, \frac{1}{2} \right)$ and, for the holomorphic map $\zeta \mapsto \zeta a$ of $\Lambda$ into $\ell^\infty$, $Sp \zeta a = \Delta \left( \frac{|\zeta|}{3} \right)$. For any $\zeta \in \Delta$

\[ h_d (Sp \zeta a, Sp b) = \max \left( \frac{1}{4}, \frac{1}{2} - \frac{|\zeta|}{3} \right) . \]

Thus,

\[ h_d (Sp \zeta a, Sp b) = \frac{1}{2} - \frac{|\zeta|}{3} \quad \text{if} \quad |\zeta| \leq \frac{3}{4} , \]

\[ h_d (Sp \zeta a, Sp b) = \frac{1}{4} \quad \text{if} \quad \frac{3}{4} \leq |\zeta| < 1 . \]

Hence the continuous non-constant function $\zeta \mapsto h_d (Sp \zeta a, Sp b)$, which reaches its maximum on $\Delta$ at $\zeta = 0$, is not subharmonic on $\Delta$.

2. The above example will now be suitably modified so as to provide an example of a non-subharmonic function defined in terms of $h_d$.

If the domain $D$ is the unit disc $\Delta$, the Carathéodory distance $c$ coincides with the Poincaré distance

\[ \omega (z_1, z_2) = \frac{1}{2} \log \frac{1 + |z_1, z_2|}{1 - |z_1, z_2|} , \]

where

\[ |z_1, z_2| = \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right| (z_1, z_2) \in \Delta . \]

It is well known that the function $(z_1, z_2) \mapsto |z_1, z_2|$ defines a distance on $\Delta$, and it is easily checked that $\omega (z_1, z_2)$ is a strictly increasing function of $|z_1, z_2|$ for $z_1$ and $z_2$ in $\Delta$. The geodesic lines through the center $0$ of $\Delta$ for the Poincaré metric are the radii of the disc $\Delta$. These facts, coupled with the invariance of $Sp b$ under rotations around $0$, imply that, for $|z| \leq \frac{1}{4}$,
the distance \([z, \text{Sp } b] = [\text{Sp } b, z] = \inf \{[z, u] : u \in \text{Sp } b\}\) between \(z\) and \(\text{Sp } b\) is given by

\[
[z, \text{Sp } b] = \left[|z|, \frac{1}{4}\right].
\]

Therefore, for \(\zeta \in \Delta\),

\[
(2) \quad \sup \{[z, \text{Sp } b] : z \in \text{Sp } \zeta a\} = \left[0, \frac{1}{4}\right] = \frac{1}{4}.
\]

Similarly, for \(\zeta \in \Delta\),

\[
(3) \quad \sup \{[z, \text{Sp } \zeta a] : z \in \text{Sp } b\} = \left[\frac{|\zeta|}{3}, \frac{1}{2}\right] = \frac{3 - 2|\zeta|}{6 - |\zeta|}.
\]

Comparison of (2) and (3) shows that

\[
h_e(\text{Sp } \zeta a, \text{Sp } b) = \frac{3 - 2|\zeta|}{6 - |\zeta|} \quad \text{if} \quad |\zeta| \leq \frac{6}{7},
\]

\[
h_e(\text{Sp } \zeta a, \text{Sp } b) = \frac{1}{4} \quad \text{if} \quad \frac{6}{7} \leq |\zeta| < 1.
\]

Hence the continuous, non-constant function \(\zeta \mapsto h_e(\text{Sp } \zeta a, \text{Sp } b)\) on \(\Delta\), which reaches its maximum at \(\zeta = 0\), is not subharmonic.

3. Examples of holomorphic maps into a non-commutative Banach algebra will now be constructed, for which the functions (1) are not upper semicontinuous.

For any \(p\) such that \(1 \leq p < \infty\), let \(l^p(\pm \infty)\) be the complex Banach space of all bilateral sequences \(x = \{x_n\}\) of complex numbers \(x_n (n \in \mathbb{Z})\) with the norm

\[
\|x\| = (\sum |x_n|^p)^{1/p}.
\]

Let \(A\) be the unital complex Banach algebra of all bounded linear operators on \(l^p(\pm \infty)\), and let \(T, S\) be the elements of \(A\) defined on the canonical basis \(\{e_n\}\) of \(l^p(\pm \infty)\) by

\[
T e_0 = 0, \quad T e_n = e_{n-1} \quad \text{for all} \quad n \neq 0,
\]

\[
S e_0 = e_{-1}, \quad S e_n = 0 \quad \text{for all} \quad n \neq 0.
\]

Let \(F : \mathbb{C} \to A\) be defined by \(F(\zeta) = T + \zeta S\). Then [1; p. 210] \(\text{Sp } F(0) = \overline{\Delta}\), \(\text{Sp } F(\zeta) = \delta \Delta = \{z \in \mathbb{C} : |z| = 1\}\) if \(\zeta \neq 0\). Therefore

\[
h_d(\text{Sp } F(\zeta), \text{Sp } T) = 1 \quad \text{for all} \quad \zeta \neq 0,
\]

\[
h_d(\text{Sp } F(0), \text{Sp } T) = 0.
\]
That shows that \( \zeta \mapsto h_d(\text{Sp } F(\zeta), \text{Sp } T) \) is not upper semicontinuous. The above example can be adapted to the Poincaré distance on \( \Delta \). Choose

\[
G(\zeta) = \frac{1}{2} F(\zeta).
\]

Then

\[
h_e(\text{Sp } G(\zeta), \text{Sp } G(0)) = \omega(0, \frac{1}{2})
\]

whenever \( \zeta \neq 0 \), showing that \( \zeta \mapsto h_e(\text{Sp } G(\zeta), \text{Sp } G(0)) \) is not upper semicontinuous.

BIBLIOGRAFIA