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Analisi matematica. — *Some Characterization of the* q-*Gamma Function by Functional Equations.* Nota I di MARINO BADIALE, presentata (*) dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — In questo lavoro, suddiviso in una Nota I e in una Nota II, si estendono alle funzioni q-gamma i classici risultati sulla determinazione univoca della funzione gamma tramite equazioni funzionali; si introduce poi una q-generalizzazione di una funzione fattoriale intera, e se ne indicano le principali proprietà.

The q-gamma functions are defined by

$$\Gamma_q(x) = \begin{cases} \dfrac{(q\,;q)_\infty (1-q)^{1-x}}{(q^x\,;q)_\infty} & \text{if } 0 < |q| < 1 \\[2ex] \dfrac{q^{(x/2)} (q^{-1}\,;q^{-1})_\infty (q-1)^{1-x}}{(q^{-x}\,;q^{-1})_\infty} & \text{if } |q| > 1 \end{cases}$$

where $(a\,;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$ and q and x are real or complex variables.

This family of functions provides a natural generalization of the classical gamma function, which may be considered the case $q = 1$. They were introduced by F. H. Jackson [1], and have recently been the object of renewed interest: see Askey [2 and 3], Moak [4] and, for the p-adic analogue, Koblitz [5 and 6]. It is known that the q-gamma functions satisfy suitable analogues of many of the classical functional equations satisfied by $\Gamma(x)$, and it is therefore natural to ask, in analogy with chapter 6 of E. Artin's celebrated monograph [7], how the functions $\Gamma_q(x)$ are characterized by their functional equations. The usual gamma function satisfies the following well-known functional equations:

(1) $$f(x+1) = xf(x) \qquad \text{(factorial property)}$$

(2) $$f(x/p)\, f((x+1)/p)\,,\cdots,\, f((x+p-1)/p) = [(2\pi)^{(p-1)\frac{1}{2}}/p^{x-\frac{1}{2}}]\, f(x)$$

this is Gauss multiplication formula, with the special case $p = 2$

(3) $$f(x/2)\, f((x+1)/2) = \frac{\sqrt{\pi}}{2^{x-1}} f(x) \qquad \text{(Legendre duplication formula).}$$

In [7] Artin proved three distinct theorems characterizing $\Gamma(x)$

a) $\Gamma(x)$ is the unique function satisfying (1) and (3) which is positive for positive x and which has continuous second derivative;

(*) Nella seduta dell'8 gennaio 1983.

b) $\Gamma(x)$ is the unique function which satisfies (1) and (2) for some integer p, is positive for positive x and which has continuous first derivative;

c) $\Gamma(x)$ is the unique function which satisfies (1) and (2) for all positive integers p, is positive for positive x, and which is continuous.

All of these results are obtained considering $\Gamma(x)$ as a real valued function of a real variable. One notes that while *a*) and *b*) are rather similar (indeed, *b*) implies *a*)), passing from *b*) to *c*) involves on the one hand a notable weakening of the 'regularity' condition (simple continuity rather than continuity of the derivative), and on the other hand a strengthening of the 'functional equation condition' ((2) must hold for all p rather than for just one). Furthemore, *a*) and *b*) are established by completely elementary arguments, whereas *c*) requires use of simple properties of Fourier series.

In what follows we will extend these results to the q-gamma functions, seeking to mimic the elementary methods of Artin as far as possible. As might be expected, the passage to functions of two variables will require the addition of certain natural 'boundary conditions' to insure uniqueness.

1. We begin by reviewing the q-analogues of properties (1)-(3) of the gamma functions. We note first that the function $\Gamma_q(x)$ interpolates the q-factorial $n!_q = (1+q)\cdots(1+q+\cdots+q^{n-1})$ just as $\Gamma(x)$ interpolates the usual factorial: $\Gamma_q(n+1) = n!_q$. Indeed $\Gamma_q(1) = 1$ for all q, and furthemore one has:

$$\text{(i)} \quad \Gamma_q(x+1) = \frac{1-q^x}{1-q}\,\Gamma_q(x) \qquad (q\text{-factorial property})$$

$$\text{(ii)} \quad \Gamma_q(nx)\,\Gamma_{q^n}(1/n)\,\Gamma_{q^n}(2/n),\cdots,\Gamma_{q^n}((n-1)/n) =$$
$$= \Gamma_{q^n}(x)\,\Gamma_{q^n}(x+1/n),\cdots,\Gamma_{q^n}(x+(n-1)/n)\,(1+q+\cdots+q^{n-1})^{nx-1}$$

(Multiplication formula)

$$\text{(iii)} \quad \Gamma_q(2x)\,\Gamma_{q^2}(1/2) = \Gamma_{q^2}(x)\,\Gamma_{q^2}(x+1/2)\,(1+q)^{2x-1}$$

(Duplication formula).

Properties (i)-(iii) although apparently more forbidding than the corresponding functional equations of $\Gamma(x)$ are, in fact, easier to prove, being consequences of the definition of $\Gamma_q(x)$ and simple algebraic manipulations. Moreover, since $\Gamma_q(x) \to \Gamma(x)$ when $q \to 1$ through real values, it is easy to convince oneself that these functional equations tend to the corresponding equations for $\Gamma(x)$ when $q \to 1$, and so they represent natural extension of classical formulae. In the sequel we shall always interpret functional equation of $\Gamma_q(x)$ in this way for $q = 1$. One notes, however, that in (ii) and (iii) the functions which appear on the two sides of the equation involve transformation of the parameter q as well as the variable x, in contradistinction to the classical Gauss multiplication formula (which corresponds to $q = q^n = 1$ in (ii)).

PROPOSITION 1: *Let $f(q, x)$ be a real valued function of two real variables with $q > 0$ and $x > 0$ such that $d^2 f/dx^2$ exists for all q and all x, and suppose that $f(q, x)$ satisfies:*

(1.1) $$f(q, x+1) = \frac{1-q^x}{1-q} f(q, x) \qquad \textit{for all } q \textit{ and all } x$$

(1.2) $$f(q, 2x) f(q^2, 1/2) = f(q^2, x) f(q^2, x+1/2)(1+q)^{2x-1} \qquad \textit{for all } q \textit{ and all } x$$

(1.3) $$f(q, x) > 0 \qquad \textit{for all } q \textit{ and all } x$$

(1.4) $$\lim_{q\to 1} f(q, x_0) = \Gamma(x_0) \qquad \textit{for some } x_0 > 0$$

(1.5) *the function $g(q, x) = d^2/dx^2 (\log(f(q, x)/\Gamma_q(x)))$ is bounded on the domain of $f(q, x)$.*

Then $f(q, x) = \Gamma_q(x)$ for all q and all x.

Proof. Let $\varphi(q, x) = f(q, x)/\Gamma_q(x)$. One then has

(1.6) $$\varphi(q, x) = \varphi(q, x+1)$$

(1.7) $$\varphi(q, x)\varphi(q^2, 1/2) = \varphi(q^2, x/2)\varphi(q^2, (x+1)/2).$$

Equation (1.6) permits us to extend the domain of $\varphi(q, x)$ to all real values of x maintaining the assumed differentiability properties, and we shall always consider $\varphi(q, x)$ to be so extended in the sequel. To prove the proposition we must show that $\varphi(q, x) \equiv 1$, and by periodicity in x we may assume $0 \leq x \leq 1$. Taking the logarithm of (1.7) and differentiating twice with respect to x, one obtains

$$g(q, x) = [g(q^2, x/2) + g(q^2, (x+1)/2] \, 1/4.$$

Thus, if M is the bound for $g(q, x)$ assumed in (1.5), one finds $|g(q, x)| \leq \leq |g(q^2, x/2)| \, 1/4 + |g(q^2, (x+1)/2)| \, 1/4 \leq M/4 + M/4 = M/2$ for $q > 0$ and $0 \leq x \leq 1$. Iterating this argument one shows that $|g(q, x)| \leq M/2^n$ for all positive integers n and so in fact $g(q, x) = 0$. Hence $\log\varphi(q, x) = a_q x + b_q$ where a_q and b_q are constants depending only on q. However, (1.6) implies that $\log\varphi(q, x)$ is periodic in x with period 1, whence $\log\varphi(q, 0) = \log\varphi(q, 1)$ and so $a_q + b_q = b_q$. Hence $a_q = 0$ and $\log\varphi(q, x)$ is constant with respect to x, as is $\varphi(q, x)$ which we may now write $\varphi(q)$. Equation (1.7) now says $\varphi(q)\varphi(q^2) = \varphi(q^2)^2$ and (1.3) (together with the analogous property for $\Gamma_q(x)$) then implies that $\varphi(q) = \varphi(q^2)$. This holds for all q, hence we may iterate to conclude $\varphi(q) = \varphi(q^{1/2}) = \cdots = \varphi(q^{1/2^n}) = , \cdots, \lim_{q\to 1}\varphi(q) = 1$ by (1.4) and the definition of $\Gamma_q(x)$. Thus $\varphi(q, x) \equiv 1$. QED.

We note that the proof shows that the 'boundary conditions' (1.4) may be weakened if we are willing to accept a scalar multiple of $\Gamma_q(x)$ rather than $\Gamma_q(x)$ itself. In this case, it suffices to require only the existence of a nonzero limit for $f(q, x_0)$ as $q \to 1$. Also, it is clear that condition (1.4) could be replaced by: (1.4)' $\lim_{q \to 0^+} f(q, x_0) = 1$ for some $x_0 > 0$ or by (1.4)'' $\lim_{q \to \infty} f(q, x_0)/\Gamma_q(x_0) = 1$ for some $x_0 > 0$, or by requiring only one of the one-sided limits in (1.4).

Hence the proof of proposition 1 is valid also if the domain of $f(q, x)$ is given by $x > 0, 0 < q < 1$ or $x > 0$ and $q > 1$.

In this way proposition 1 may be interpreted as separate characterizations of $\Gamma_q(x)$ for $0 < q < 1$ and for $q > 1$.

The following corollary is now evident. It is an analogue of theorem a).

COROLLARY: *Let $f(q, x)$ be a positive continuous real-valued function for $0 \le q \le 1$ and $x > 0$ such that $\mathrm{d}^2 f/\mathrm{d}x^2$ is continuous in both q and x. Suppose that $f(q, x)$ satisfies (1.1) and (1.2) and that $f(0, x_0) = 1$ for some x_0. Then $f(q, x) = \Gamma_q(x)$. (For $q = 1$ equation (1.1) should be interpreted as equation 1).*

In view of theorem b) it is now reasonable to ask if the differentiability hypothesis may be weakened, for example, by requiring only that $\mathrm{d}f/\mathrm{d}x$ be continuous. If we take as the domain of $f(q, x)$ the open set $0 < q < 1, 0 < x$ as in proposition 1 (and not the domain of $f(q, x)$ in the corollary which includes the points with $q = 0$), then the following counterexample shows that the boundedness of second derivative is essential: Let $h(q, x) = (\log q)^2 \sum_{n=1}^{\infty} 2^{-3n} \sin(2^n \pi x)$ and set $f(q, x) = \Gamma_q(x) \cdot \exp(h(q, x))$. It is easy to see that the series for $h(q, x)$, $\mathrm{d}h/\mathrm{d}x$, and $\mathrm{d}^2 h/\mathrm{d}x^2$ all converge uniformly in x for each fixed $q, 0 < q < 1$, and that $h(q, x+1) \neq h(q, x)$. It follows that $f(q, x)$ satisfies (1.1) and (1.3). Moreover, we have $h(q, 1/4) \neq 0$, so $h(q, x) \neq 0$; but we have $h(q, x) + h(q^2, 1/2) = h(q^2, x/2) + h(q^2, (x+1)/2)$: indeed, since $h(q^2, 1/2) = 0$, it suffices to show $h(q, x) = h(q^2, x/2) + h(q^2, (x+1)/2)$ and we have

$$
\begin{aligned}
h(q^2, x/2) + h(q^2, (x+1)/2) &= (\log q^2)^2 \sum_{n=1}^{\infty} 2^{-3n} \sin(2^n \pi x/2) + \\
&+ (\log q^2)^2 \sum_{n=1}^{\infty} 2^{-3n} \sin(2^n \pi (x+1)/2) = \\
&= 4 (\log q)^2 \sum_{n=1}^{\infty} 2^{-3n} \sin(2^{n-1} \pi x) + \\
&+ 4 (\log q)^2 \sum_{n=1}^{\infty} 2^{-3n} \sin(2^{n-1} \pi x + 2^{n-1} \pi) = \\
&= (\log q)^2 \sum_{n=1}^{\infty} 2^{-3n+2} \sin(2^{n-1} \pi x) +
\end{aligned}
$$

$$+ (\log q)^2 \sum_{n=1}^{\infty} 2^{-3n+2} \sin (2^{n-1} \pi x + 2^{n-1} \pi) =$$

$$= (\log q)^2 \sum_{n=2}^{\infty} 2^{-3n+3} \sin (2^{n-1} \pi x) + \tfrac{1}{2} (\log q)^2 \sin \pi x +$$

$$+ (\log q)^2 \tfrac{1}{2} \sin (\pi x + \pi) =$$

$$= (\log q)^2 \sum_{m=1}^{\infty} 2^{-3m} \sin 2^m \pi x) = h(q, x) .$$

It follows that $f(q, x)$ satisfies (1.2). Finally, $g(q, x)$ satisfies (1.4) for all x_0. Thus $f(q, x)$ satisfies all the conditions of proposition 1 except tha boundedness of $d^2 f/dx^2$, which is a continuous function with non compact domain. Note also that $f(q, x)$ satisfies (1.4)′ for $x_0 = 1/2$.

We note in passing that the function $f(q, x) = q^{(\sin(2\pi x)/4\pi^2)} \Gamma_q(x)$, introduced by Moak [4] for $q > 1$ as a counterexample to the 'obvious' generalization of the Bohr-Mollerup theorem to these $\Gamma_q(x)$, satisfies neither (1.2) nor (1.5), although it does meet requirements (1.1), (1.3) and (1.4). A slight modification of Moak's function provides an example of a function defined for $0 \leq q \leq 1$ and $x > 0$ and continuous there, and which satisfies all the conditions of proposition 1 except (1.2): Let $h_q(x) = q^{4+\sin 2\pi x} - 1$ and set $f(q, x) = e^{h_q(x)} \Gamma_q(x)$.

Then (1.1) follows from periodicity of $h_q(x)$ in x; (1.3) is obvious; (1.4) holds for all x since $h_q(x) \to 0$ as $q \to 1$; and (1.5) follows from a simple calculation (in fact, $h_q''(x) \to 0$ as $q \to 0$ uniformly with respect to x, which allows us to extend the domain of $h_q''(x)$ to include the points with $q = 0$). If we replace 4 by -4 in the definition of $h_q(x)$ we obtain a counter-example with the same characteristics in the domain $q \geq 1, x > 0$.

Here we conclude the first part of our investigation. For the bibliography see our subsequent article.

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