Characterization of some interpolation spaces (II)

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<http://www.bdim.eu/item?id=RLINA_1982_8_72_6_333_0>

RIASSUNTO. — Si caratterizzano alcuni spazi di interpolazione tra spazi di funzioni continue e domini di operatori ellittici del 2° ordine.

INTRODUCTION

In the study of evolution equations, it is sometimes useful to work in some interpolation space between the domain $D(A)$ of a differential operator $A$ and the function space $X$ in which $D(A)$ is embedded (see for instance Da Prato-Grisvard [1], Sinestrari [4], [5], Da Prato-Sinestrari [2]). Our kind of interpolation spaces are defined as follows: if $A : D(A) \subset X \to X$ is the infinitesimal generator of a semigroup $e^{tA}$ in $X$, for every $0 \in ]0, 1[, \text{ set:}$

$$D_A(\theta) = \{ x \in X ; \lim_{t \to 0^+} t^{-\theta} (e^{tA}x - x) = 0 \} .$$

In this work we characterize $D_A(\theta)$ when $X$ is a space of continuous functions and $A$ is a strongly elliptic operator. More precisely, let $\Omega$ be an open set of $\mathbb{R}^n$ with regular boundary $\partial \Omega$. Then it is known (see Stewart [6]) that every strongly elliptic operator $A$ with sufficiently regular coefficients is the infinitesimal generator of an analytic semigroup in $C_0(\Omega)$, the space of all continuous functions in $\Omega$ which vanish on $\partial \Omega$. We prove that, if $\theta \neq \frac{1}{2}$, then

$$D_A(\theta) = H^{2\theta}(\Omega) \cap C_0^0(\Omega),$$

where $H^{2\theta}(\Omega)$ is the space of little H"older continuous functions of order $2\theta$. If $2\theta < 1$, then $H^{2\theta}(\Omega)$ is defined as the subspace of $C^{2\theta}(\Omega)$ consisting of all $f$ such that

$$\lim_{\tau \to 0^+} \sup_{x, y \in \Omega, \|x - y\| \leq \tau} \frac{|f(x) - f(y)|}{\tau^{2\theta}} = 0,$$

whereas, if $2\theta > 1$, $H^{2\theta}(\Omega)$ is defined as the space of all $f \in C^2(\Omega) \cap C_0^0(\Omega)$ such that $\frac{\partial^2 f}{\partial x_i^2} \in H^{2\theta-1}(\Omega) \quad \forall i = 1, \ldots, n$. A similar result holds if we take $X = C^0(\Omega)$ (without requiring boundary conditions).


1. Definitions and Some Properties of Interpolation Spaces

Throughout this section X and Y will denote two Banach spaces, with Y continuously embedded in X (we shall write \( Y \hookrightarrow X \)).

**Definition 1.1.** For every \( \theta \in [0,1] \) set:

\[
C(\theta\ , Y, X) = \{ u : [0,1] \to X \ ; \ t \to t^\theta u(t) \in C([0,1] ; Y), \ t \to t^\theta u'(t) \in C([0,1] ; X), \lim_{t \to 0^+} \| t^\theta u(t) \|_Y = \lim_{t \to 0^+} \| t^\theta u'(t) \|_X = 0 \} .
\]

\( C(\theta\ , Y, X) \) is a Banach space under the norm:

\[
\| u \|_{0; Y, X} = \| t^\theta u(t) \|_{L^\infty(0,1; Y)} + \| t^\theta u'(t) \|_{L^\infty(0,1; X)} .
\]

It is easy to show that if \( u \in C(\theta\ ; Y, X) \) then there exists

\[
X - \lim_{t \to 0^+} u(t) = u(0) .
\]

**Definition 1.2.** For every \( \theta \in [0,1] \) set:

\[
(Y, X)_0 = \{ u(0) , u \in C(\theta\ ; Y, X) \} .
\]

\( (Y, X)_0 \) is a Banach space under the norm:

\[
\| x \|_0 = \inf_{u \in C(\theta; Y, X)} \| u \|_{0; Y, X} ,
\]

and it can be shown that Y is dense in \( (Y, X)_0 \) (see Sinestrari [3]).

In Section 2 we shall also use the following characterization of \( (Y, X)_0 \):

**Proposition 1.3.** \( (Y, X)_0 \) is the set of all \( x \in X \) such that:

\[
x = u(t) + v(t) \quad \forall t \in [0,1]
\]

where:

\[
(1-2) \quad \left\{ \begin{array}{c}
t \to t^\theta u(t) \in C([0,1] ; Y) \\
t \to t^{\theta-1} v(t) \in C([0,1] ; X)
\end{array} \right.
\]

and the norm:

\[
\| x \| = \inf_{u, v \text{ satisify (1-2)}} (\| t^\theta u \|_{L^\infty(0,1; Y)} + \| t^{\theta-1} v \|_{L^\infty(0,1; X)})
\]

is equivalent to the norm of \( (Y, X)_0 \).

In the case \( Y = D(A) \), where A is the infinitesimal generator of a semigroup \( e^{tA} \) in X, there are other useful characterizations of \( (Y, X)_0 \). Define,
for every $\theta \in [0, 1[$:

$$D_A(\theta) = \{ x \in X \ ; \lim_{t \to 0^+} t^{-\theta} (e^{tA} x - x) = 0 \},$$

$$\| x \|_{D_A(\theta)} = \| x \|_X + \| t^{-\theta} (e^{tA} x - x) \|_{L^\infty(0, 1; X)}.$$

Then $D_A(\theta)$ is a Banach space under the norm $\| \cdot \|_{D_A(\theta)}$ and the following characterization holds:

**Proposition 1.4.** Under the above assumptions, for every $\theta \in [0, 1[$ we have:

$$D_A(\theta) \cong (D(A), X)_{1-\theta}$$

if $D(A)$ is endowed with the graph norm.

The proofs of Propositions 1.3 and 1.4 can be found in Da Prato–Grisvard [1].

2. Characterization of some interpolation spaces

Let $\Omega$ be open in $\mathbb{R}^n$ and set:

$$C^0_\sigma(\Omega) = \{ f \in C^0(\Omega) \ ; \ f(x) = 0 \ \forall x \in \partial \Omega, \ \lim_{|x| \to +\infty} f(x) = 0 \text{ if } \Omega \text{ is unbounded}, \}$$

$$C^\sigma(\Omega) = \{ f \in C^\sigma(\Omega) \ ; \ \forall |z| = 0, 1, 2, D^\sigma f \in C^0(\Omega) \}.$$

These spaces are endowed with their natural norms.

The principal result of this paper is the characterization of the interpolation space $D_A(\theta)$ between $C^0(\Omega)$ (resp. $C^0(\Omega)$) and the domain of an elliptic operator $A$ in $C^0(\Omega)$ (resp. $C^0_\sigma(\Omega)$). To this purpose we must first consider the cases $\Omega = \mathbb{R}^n$ and $\Omega = \mathbb{R}^n_+ = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n ; x_n > 0 \}$ and characterize the interpolation spaces between $C^0(\Omega)$ and $C^2(\Omega)$ (resp. $C^0_\sigma(\Omega)$ and $C^2_\sigma(\Omega)$). We start by giving the following definition:

**Definition 2.1.** For every $\sigma \in [0, 1[$ set:

$$h^\sigma(\Omega) = \{ f \in C^0(\Omega) \ ; \ \lim_{|x| \to 0^+} \sup_{\frac{y}{z} \in \partial \Omega, |z-y| \leq r} \tau^{-\sigma} | f(x) - f(y) | = 0 \},$$

$$h^{\sigma+1}(\Omega) = \left\{ f \in C^1(\Omega) \ ; \ \frac{\partial f}{\partial x_i} \in h^\sigma(\Omega) \ \forall i = 1, \ldots, n \right\},$$

$$h^\sigma_0(\Omega) = h^\sigma(\Omega) \cap C^0_\sigma(\Omega),$$

$$h^{\sigma+1}_0(\Omega) = \left\{ f \in C^1(\Omega) \cap C^0_\sigma(\Omega) ; \ \frac{\partial f}{\partial x_i} \in h^\sigma_0(\Omega) \ \forall i = 1, \ldots, n \right\}.$$

Then $h^\sigma(\Omega)$ and $h^\sigma_0(\Omega)$ (resp. $h^{\sigma+1}(\Omega)$ and $h^{\sigma+1}_0(\Omega)$) are Banach spaces under the $C^\sigma$-norm (resp. under the $C^{\sigma+1}$-norm).
Proposition 2.2. For every $0 \in ]0, 1[, \theta \neq \frac{1}{2}$, we have:

$$\left( C^2_0(\mathbb{R}^n), C^0_0(\mathbb{R}^n) \right)_{1-\theta} \approx h^{2\theta}_0(\mathbb{R}^n),$$

$$\left( C^2_0(\mathbb{R}^n_+), C^0_0(\mathbb{R}^n_+) \right)_{1-\theta} \approx h^{2\theta}_0(\mathbb{R}^n_+).$$

Sketch of the proof: First we characterize $C^2_0(\mathbb{R}^n)$ (resp. $C^2_0(\mathbb{R}^n_+)\right)$ as the intersection of the domains of simple differential operators (the second derivatives) and then we use Proposition 1, p. 88 of Triebel [7], which can be easily adapted to our situation.

Proposition 2.3. Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ with $\partial \Omega$ of class $C^2$. Then for every $0 \in ]0, 1[, \theta \neq \frac{1}{2}$, we have:

$$\left( C^2(\overline{\Omega}) \cap C^0(\overline{\Omega}), C^0(\overline{\Omega}) \right)_{1-\theta} \approx h^{2\theta}(\overline{\Omega}) \cap C^0(\overline{\Omega}),$$

$$\left( C^2(\mathbb{R}^n), C^0(\mathbb{R}^n) \right)_{1-\theta} \approx h^{2\theta}(\mathbb{R}^n).$$

The proof is based on a method of localization which uses Proposition 2.2.

Under the hypotheses of Proposition 2.3, suppose that $\partial \Omega$ is of class $C^{2+\mu}$ for some $\mu > 0$ and set:

$$X = C^0(\overline{\Omega}),$$

$$D(A) = \{ f \in C^0(\overline{\Omega}) ; Af \in C^0(\overline{\Omega}) \},$$

$$Af = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial f}{\partial x_i} + cf$$

$\forall f \in D(A),$

$$X_0 = C^0_0(\overline{\Omega}),$$

$$D(A_0) = \{ f \in C^0(\overline{\Omega}) ; Af \in C^0(\overline{\Omega}) \},$$

$$A_0 f = Af$$

$\forall f \in D(A_0),$ where $a_{ij}, b_i, c \in C^0(\overline{\Omega})$ and there exists $\nu > 0$ such that $\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \nu |\xi|^2 \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n.$

Under all these assumptions, $A : D(A) \to X$ is the infinitesimal generator of an analytic semigroup in $X$ and $A_0 : D(A_0) \to X_0$ is the infinitesimal generator of a strongly continuous analytic semigroup in $X_0$ (see Stewart [6]).

The main result of this paper is the following:

Theorem 2.4. Under the above assumptions, for every $0 \in ]0, 1[, \theta \neq \frac{1}{2}$, we have:

$$D_A(0) \approx (D(A), C^0(\overline{\Omega}))_{1-\theta} \approx h^{2\theta}(\overline{\Omega}) \cap C^0(\overline{\Omega}),$$

$$D_{A_0}(0) \approx (D(A_0), C^0_0(\overline{\Omega}))_{1-\theta} \approx h^{2\theta}(\overline{\Omega}) \cap C^0_0(\overline{\Omega}).$$
Here we sketch the proof of (2.4). To prove \( \subset \), it is sufficient to observe that \( C^2(\Omega) \cap C^0(\Omega) \subset D(A) \), and hence we have:

\[
C^0(\Omega) \cap H^{20}(\Omega) \cong (C^2(\Omega) \cap C^0(\Omega), C^0(\Omega))_{1-\theta} \subset (D(A), C^0(\Omega))_{1-\theta} = D_A(0).
\]

Now we prove the inclusion \( \supset \). Let \( f \in D_A(0) \), then \( f = u(t) + v(t) \quad \forall t \in [0, 1] \) where \( u \) and \( v \) satisfy (1.2). For every \( t \in [0, 1] \) there exists an extension \( U(t) \) of \( u(t) \) to \( \mathbb{R}^n \) and an extension \( V(t) \) to \( \mathbb{R}^n \) such that \( U \) and \( V \) satisfy (1.2)

with \( X = C^0_0(\mathbb{R}^n), Y = C^1_0(\mathbb{R}^n) = \left\{ f \in C^1(\mathbb{R}^n); \lim_{|x| \to +\infty} \frac{f(x)}{x} = 0 \right\} \)

and moreover \( U(t) + V(t) \) is constant. Extend \( f \) to \( \mathbb{R}^n \), setting:

\[
F(x) = U(t)(x) + V(t)(x) \quad \forall x \in \mathbb{R}^n, \forall t \in [0, 1].
\]

For every \( t \in [0, 1] \) let \( Q_t \) be the cube centered at \( 0 \in \mathbb{R}^n \) with edge \( t \), and set:

\[
M_t(\varphi)(x) = \int_{x + Q_t} \varphi(y) dy \quad \forall \varphi \in C^1(\mathbb{R}^n).
\]

Then, setting:

\[
W(t)(x) = M_t(U(t))(x) \quad \forall x \in \mathbb{R}^n, \forall t \in [0, 1]
\]

we can write \( F \) as:

\[
F = W(t) + V(t) + (U(t) - W(t)) = W(t) + G(t) \quad \forall t \in [0, 1]
\]

where \( W \) and \( G \) satisfy:

\[
t \to t^{1-\theta} W(t) \in C([0, 1]; C^0_0(\mathbb{R}^n)),
\]

\[
t \to t^{1-\theta} G(t) \in C([0, 1]; C^0_0(\mathbb{R}^n)).
\]

Then \( F \) belongs to \( (C^0_0(\mathbb{R}^n), C^0_0(\mathbb{R}^n))_{1-\theta} = H^{20}(\Omega) \), and hence \( F \) is in \( H^{20}(\Omega) \). Moreover, as \( D(A) \) is dense in \( D_A(0) \), we have: \( f(x) = 0 \quad \forall x \in \partial \Omega \), so \( \subset \) of (2.4) is proved. The proof of (2.5) is analogous.
REFERENCES


