Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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The Bernoullian of a Matrix. (A Generalization of the Bernoulli Numbers)

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **72** (1982), n.6, p. 315–317.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1982_8_72_6_315_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1982.

Teoria dei numeri. — *The Bernoullian of a Matrix. (A Generalization of the Bernoulli Numbers).* Nota di ESAYAS GEORGE KUNDERT, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si associano ad una matrice infinita di un certo tipo altre due matrici dello stesso tipo, dette rispettivamente bernoulliana e antibernoulliana di A. Si studiano alcune proprietà di queste matrici. Si ottiene in tal via una generalizzazione dei classici numeri di Bernoulli.

In the following article we associate to a certain type of infinite matrix A, two other matrices of the same type, which we will call the Bernoullian, respectively the Anti-Bernoullian of A. We study some of the properties of these mappings. For example, if $A = E$ (identity matrix) then the first column of the Bernoullian E^b consists of the doubled classical Bernoulli numbers [1]. The doubled k-th column we will call the Bernoulli numbers of the k-th kind and it seems that even in this special case, these numbers were never introduced before?

Let $\{a_n\}$ be a basis of $\hat{\mathfrak{A}}$ corresponding to the operator A (see [2]). Let $a = \sum_{n=0}^{\infty} \alpha_n a_n$ and let $\overline{a} = \sum_{m=0}^{\infty} \alpha_{2m} a_{2m}$ and $\overline{\overline{a}} = \sum_{m=0}^{\infty} \alpha_{2m+1} a_{2m+1}$ so that $a = \overline{a} + \overline{\overline{a}}$. Let $\overline{\mathfrak{A}}$ be the subspace spanned by $\{a_{2m}\}$ and $\overline{\overline{\mathfrak{A}}}$ the subspace spanned by $\{a_{2m+1}\}$. $\overline{a}$ and $\overline{\overline{a}}$ are then the projections of a onto $\overline{\mathfrak{A}}$ respectively $\overline{\overline{\mathfrak{A}}}$. Let $\{a_n'\}$ be the basis corresponding to the operator $A' = A - E$ and consider the linear transformation ${}_{A}T_{A'} : a = \sum_{n=0}^{\infty} \alpha_n a_n \rightarrow a' = \sum_{n=0}^{\infty} \alpha_n a_n'$ (see [2]). Let ${}_{A}F_{A'}$ be the linear subspace of invariants under ${}_{A}T_{A'}$. We know by [2] that if $a \in {}_{A}F_{A'}$ we can prescribe $\overline{a}$ and via formula (F) of that paper, find uniquely $\overline{\overline{a}}$. The mapping $\beta : \overline{a} \rightarrow \overline{\overline{a}}$ from $\overline{\mathfrak{A}}$ onto $\overline{\overline{\mathfrak{A}}}$ is easily seen to be linear and onto.

Let $\overline{a}_{2m} = \sum_{m=0}^{\infty} \overline{\alpha}_{mk} a_{2k}$ be another basis of $\overline{\mathfrak{A}}$. Then $\overline{\overline{a}}_{2m} = \beta(\overline{a}_{2m}) = \sum_{k=0}^{\infty} \overline{\overline{\alpha}}_{mk} a_{2k+1}$ and we have a linear mapping b of the matrice $A = (\overline{\alpha}_{km}) \rightarrow A^b = (\overline{\overline{a}}_{km})$. The matrices A being those nonsingular ones with only a finite number of nonzero row elements. We denote the space of those matrices by $\mathfrak{M}$ and the mapping $b : \mathfrak{M} \rightarrow \mathfrak{M}$ is also linear and onto, which we call the Bernoullian mapping and specifically A^b we call the Bernoullian of A, while

(*) Nella seduta del 25 giugno 1982.

$A^{b^{-1}} = b^{-1}(A)$, we will name the Anti-Bernoullian of A. Let E be the identity matrix of $\mathfrak{M}$. Then the elements of the first column of E^b turn out to be exactly the doubled classical Bernoulli numbers. We may therefore call the doubled numbers of the k-th column the Bernoulli numbers of the k-th kind. To my knowledge these numbers—which must have similar properties to those of the usual Bernoulli numbers—have never been introduced nor studied before.

We write down some properties of the Bernoullian and Anti-Bernoullian matrices:

Main property: $A^b = E^b A$.

Proof. After having discovered this relation the proof is a straightforward calculation.

From this mainproperty follow easily in steps the following properties:

(1) $A = E^b A^{b^{-1}}$

(2) $E = E^b E^{b^{-1}} \Rightarrow E^{b^{-1}} = (E^b)^{-1}$

(3) $A^{b^{-1}} = E^{b^{-1}} A$

(4) $A^{b^n} = E^{b^n} A$ for all integers n.

Note also the identity: $E^b = A^b A^{-1}$ for all A.

Further noteworthy properties are:

(5) $E^{b^n} = (E^b)^n$

(6) $(AB)^b = A^b B \Rightarrow b$ is not an isomorphism.

(7) $AE^{b^n} = [(A^{-1})^{b^{-n}}]^{-1}$.

Formula (7) follows easily from $(A^{-1})^{b^{-n}} = E^{b^{-n}} A^{-1} = (E^{b^n})^{-1} A^{-1} = (AE^{b^n})^{-1}$.

We could define ${}^bA = AE^b$ as the left Bernoullian of A and ${}^{b^{-1}}A = AE^{b^{-1}}$ as the left Anti-Bernoullian of A, however property (7) tells us that these matrices can be expressed with help of (right) Anti-Bernoullians (respectively Bernoullians) and inverses: ${}^bA = [(A^{-1})^{b^{-1}}]^{-1}$ and ${}^{b^{-1}}A = [(A^{-1})^b]^{-1}$.

We may use the above mentioned properties to derive properties for Bernoulli numbers of the k-th kind (including the classical Bernoulli numbers). For example let $a = 2 + \sum_{n=1}^{\infty} x_n$. We know from [2] that $a \in {}_D F_{D'}$ and also that $S_L^k a \in {}_D F_{D'}$ for $k = 0, 1, 2, \cdots$. The coefficients of $S_L^k a$ can easily be computed and are $2k - 1$ zeros followed by 2 and then followed by the k-th summation sequence of the sequence $(2, 1, 1, 1, 1, \cdots)$ so that $S_L a = 2x_2 + \sum_{n-3}^{\infty} nx_k$ and $S_L^2 a = 2x_4 + 5x_5 + 9x_6 + 14x_7 + 20x_8 + \cdots$. Now let $\overline{S}_L^k a$ and $\overline{\overline{S}}_L^k a$

be the projections of $S_L^k\, a$ onto $\bar{\mathfrak{A}}$ and $\bar{\bar{\mathfrak{A}}}$ respectively. The coefficient matrices are:

$$A = \begin{vmatrix} 1 \\ 1 & 3 \\ 1 & 5 & 5 \\ 1 & 7 & 14 & 7 \\ 1 & 9 & 27 & 30 & 9 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \quad \text{and} \quad A^b = \begin{vmatrix} 2 \\ 1 & 2 \\ 1 & 4 & 2 \\ 1 & 6 & 9 & 2 \\ 1 & 8 & 20 & 16 & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

we can therefore compute E^b with help of the identity $E^b = A^b A^{-1}$ where A as well as A^b are both matrices stemming from the matrix

$$\tilde{A} = \begin{vmatrix} 2 \\ 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 & 2 \\ 1 & 5 & 5 \\ 1 & 6 & 9 & 2 \\ 1 & 7 & 14 & 7 \\ 1 & 8 & 20 & 16 & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

namely A^b by taking the even numbered rows and A by taking the odd numbered rows. $\tilde{A}$ has as column vectors the vectors $S_L^k\, a$ which are computed as prescribed above.

LITERATURE

[1] P. BACHMANN (1968) – *Niedere Zahlentheorie*, 2-ter Teil, Chelsea Publishing Co.
[2] E. G. KUNDERT (1978) – *Basis in a certain Completion of the s–d–ring over the rational numbers*. Nota II, « Acc. Naz. Lincei », ser. VIII, vol. LXIV, 6.