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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On the spectral sequence for the $\bar{\partial}$-cohomology of a holomorphic bundle with Stein fibres* (*). Nota di GUIDO LUPACCIOLU, presentata (**) dal Socio E. MARTINELLI.

RIASSUNTO. — Si esamina la successione spettrale per la $\bar{\partial}$–coomologia dello spazio totale di un fibrato olomorfo nel caso in cui le fibre siano varietà di Stein.

The present paper deals with the spectral sequence for the $\bar{\partial}$-cohomology of a holomorphic fibre bundle associated with the filtration of differential forms by the lowest degree in base coordinates.

For the case of compact fibres a detailed discussion on this subject is found in A. Borel [1]. It shows that the above spectral sequence proceeds essentially in the same way as the canonical spectral sequence for ordinary cohomology of fibre bundles (1), provided every connected component of the structure Lie group of the bundle acts trivially on the $\bar{\partial}$-cohomology space of the typical fibre (2).

In this paper we consider the case when the fibres of the bundle are Stein manifolds, showing that the results of [1] are valid also in this case. The bundle need not be equipped with a complex Lie group as a structure group.

As an application, we shall use the spectral sequence to achieve a result that generalizes a well-known property of Stein manifolds.

(*) Lavoro eseguito nell'ambito del G.N.S.A.G.A.

(**) Nella seduta del 25 giugno 1982.

(1) See for example [2].

(2) This condition is automatically fulfilled if the typical fibre is compact Kählerian (see [1], 1.4).

1. Preliminaries

Let $\xi = (E, B, F, \pi)$ a locally trivial holomorphic fibre bundle, with bundle space E, base B, typical fibre F and projection π, where E, B, F are finite dimensional connected complex manifolds (3) and F is a Stein manifold.

Let W be a holomorphic complex vector bundle on B with fibre dimension n and $\hat{W} = \pi^* W$ the induced vector bundle on E by the projection $\pi : E \to B$.

As usual the space of smooth differential forms on E with coefficients in $\hat{W}$ is denoted $A_E(\hat{W}) = \Sigma A_E^{p,q}(\hat{W})$ (omitting $\hat{W}$ if the latter is the trivial one-dimensional complex vector bundle on E, *i.e.* in the case of ordinary complex-valued forms). Since $\hat{W}$ is holomorphic, $A_E(\hat{W})$ is a differential module under $\bar{\partial}$: the corresponding derived module is called the $\bar{\partial}$-cohomology space of E with coefficients in $\hat{W}$ and denoted $H_{\bar{\partial}}(E, \hat{W}) = \Sigma H^{p,q}(E, \hat{W})$.

The spectral sequence $\{E_r, d_r\}_{r \geq 0}$ to be inspected here is (as in [1]) the one abutting to $H_{\bar{\partial}}(E, \hat{W})$ which arises in the standard way from the decreasing filtration $\{F^s A_E(\hat{W})\}_{s \geq 0}$ described below.

Locally, near a point $z \in E$, a form $\omega \in A_E(\hat{W})$, after the choice of a local trivialization of $\hat{W}$, may be viewed as a n-tuple $(\omega_1, \cdots, \omega_n)$ of complex-valued forms, which may be called the *components* of ω under that local trivialization of $\hat{W}$. Assume that every ω_i, when expressed in terms of local product coordinates (x, y) (x the base coordinates), in a neighbourhood of z is the sum of monomial forms each having total degree $\geq s$ in the base coordinates. This does not depend on the choice of the local product coordinates, because a change of such coordinates is of the form $x' = x'(x)$, $y' = y'(x, y)$. Moreover, it is readily seen that the same property pertains also to the components of ω under any other local trivialization of $\hat{W}$ near z. Therefore the above assumption has an intrinsic meaning with regard to ω, which may be expressed by saying that the lowest total degree of ω in base coordinates is $\geq s$ near the point z. Then $F^s A_E(\hat{W})$ is the subspace of $A_E(\hat{W})$ containing the forms whose lowest total degree in base coordinates in $\geq s$ near each point of E. It is clear that $\{F^s A_E(\hat{W})\}_{s \geq 0}$ is a decreasing filtration stable under $\bar{\partial}$.

2. The term E_0

It is readily seen that the result obtained in [1] at this stage, expressed by the Lemma 5.1 (p. 209) is valid in general, *i.e.* independently on the assumptions under which [1] proceeds. Indeed, by a direct inspection of the corresponding proof, it appears that the compactness of F is not used anywhere and the structu-

(3) Manifolds are assumed to be Hausdorff and with a countable basis of open sets.

re Lie group plays an inessential role, because the $\psi_{\alpha\beta}$ may be thought of as transition functions with values in Aut (F) (4).

Therefore the Lemma 5.1 of [1] is also valid in particular for the case considered here.

Let us rephrase that Lemma so as to express it in a more synthetic and meaningful way, avoiding sheaves, in terms of differential forms with coefficients in a vector bundle.

It is clear that the spaces $A_{F_x}(W_x)$ $(x \in B)$ of smooth differential forms on the fibres of ξ with coefficients in the corresponding fibres of W may be viewed in a natural way as the fibres of a complex Frechet bundle on B, with the space $A_F(\mathbf{C}^n)$ as typical fibre. This bundle is denoted here $\mathbf{A}_F(W)$ (omitting W if the latter is the trivial one-dimensional complex vector bundle on B (5); $\mathbf{A}_F^{a,b}(W)$ $(a, b \geq 0)$ are the summands of $\mathbf{A}_F(W)$ is the obvious splitting and $\bar\partial_F$ is the endomorphism of $\mathbf{A}_F(W)$ induced by the differential $\bar\partial$ of $A_F(\mathbf{C}^n)$.

Starting from a system $\{\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Aut}(F)\}$ of transition functions of ξ, one obtains a system $\{\psi'_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(A_F(\mathbf{C}^n))\}$ of transition functions of $\mathbf{A}_F(W)$ simply by composing each $\psi_{\alpha\beta}$ with the natural representation of Aut(F) on $A_F(\mathbf{C}^n)$. The functions $\psi'_{\alpha\beta}$ are smooth, for if $\rho : U_\alpha \cap U_\beta \to A_F(\mathbf{C}^n)$ is smooth, so too is $\psi'_{\alpha\beta} \circ \rho : U_\alpha \cap U_\beta \to A_F(\mathbf{C}^n)$ (given by $x \mapsto [\psi'_{\alpha\beta}(x)](\rho(x))$). It follows that $\mathbf{A}_F(W)$ may be regarded as a smooth vector bundle, whence the space $A_B(\mathbf{A}_F(W))$ of smooth differential forms on B with coefficients in $\mathbf{A}_F(W)$ makes sense. Moreover it is plain that $A_B(\mathbf{A}_F(W))$ is a differential module under the endomorphism induced by $\bar\partial_F$, also denoted $\bar\partial_F$.

Then the Lemma 5.1 of [1] is equivalent to the following

PROPOSITION 2.1. *There exist canonical isomorphisms*

$$^{(p,q)}k_0^s : {}^{(p,q)}E_0^s \xrightarrow{\sim} \sum_{i \geq 0} A_B^{i,s-i}(\mathbf{A}_F^{p-i,q-s+i}(W)) \qquad (p, q, s \geq 0).$$

Moreover the sum k_0 of these carries d_0 onto $\bar\partial_F$.

Here s and (p, q) are the degree and the bidegree induced respectively by the filtering degree and by the type in $A_E(\hat{W})$.

3. THE TERM E_1

After Proposition 2.1, the discussion on the term E_1 amounts to the computation of the derived module of the differential module $(A_B(\mathbf{A}_F(W)), \bar\partial_F)$.

In order to show that the present case proceeds as that of [1] (6), we must prove that there exist exact sequences corresponding to 3.6 (3) and 3.8 (5) of [1].

(4) As usual Aut (F) denotes the group of all biholomorphic homeomorphisms of F.

(5) Clearly $\mathbf{A}_F(W)$ may be identified (by a canonical isomorphism) with the tensor product vector bundle $W \otimes \mathbf{A}_F$.

(6) See [1], § 3.

Now it is clear that the exact sequence 3.6 (3) is valid in general and that the exact sequence 3.8 (5) in any case satisfies the following conditions:

I) The spaces $H_{\bar\partial}(F_x)$ $(x \in B)$ of $\bar\partial$-cohomology of the fibres of ξ are the fibres of a smooth complex vector bundle on B, with the space $H_{\bar\partial}(F)$ as typical fibre; this bundle is denoted $\mathbf{H}_{\bar\partial}(F)$ and $\mathbf{H}^{a,b}(F)$ $(a, b \geq 0)$ are its summands in the obvious splitting.

II) The sequence of sheaves corresponding to 3.7 (4) of [1] is exact.

Therefore we need only show that I) and II) hold in the present case.

It is a well-known fact that F being a Stein manifold entails that the only non vanishing $\bar\partial$-cohomology spaces of F are the $H^{a,0}(F)$ $(a \geq 0)$, which coincide with the spaces Ω_F^a of complex-valued holomorphic a-forms on F. It follows that I) amounts only to the assertion that, for each $a \geq 0$, the spaces $\Omega_{F_x}^a$ $(x \in B)$ are the fibres of a smooth complex vector bundle on B, say $\boldsymbol{\Omega}_F^a$, with the space Ω_F^a as typical fibre. This is obvious, since $\boldsymbol{\Omega}_F^a$ is nothing but the kernel of the Frechet bundle homomorphism $\bar\partial_F : \mathbf{A}_F^{a,0} \to \mathbf{A}_F^{a,1}$.

More generally we may consider the smooth complex vector bundle on B $\mathbf{H}_{\bar\partial}(F, W)$, whose fibres are the spaces $H_{\bar\partial}(F_x, W_x)$ $(x \in B)$ (7). This is given by

$$\mathbf{H}^{a,0}(F, W) = \boldsymbol{\Omega}_F^a(W) \quad , \quad \mathbf{H}^{a,b}(F, W) = 0 \qquad (a \geq 0, b \geq 1), \tag{3.1}$$

where $\boldsymbol{\Omega}_F^a(W)$ is the kernel of $\bar\partial_F : \mathbf{A}_F^{a,0}(W) \to \mathbf{A}_F^{a,1}(W)$.

To dispose of II) we need only prove the inclusion of sheaves $\mathfrak{Z}^{a,b} \subset \bar\partial_F \mathfrak{F}^{a,b-1}$ $(a \geq 0, b \geq 1)$, since now the sheaf $\mathfrak{H}^{a,b}(F)$ on B of germs of smooth sections of $\mathbf{H}^{a,b}(F)$ is zero for $b \geq 1$. This amounts to the Poincaré lemma for the differential sheaf $(\mathfrak{F}, \bar\partial_F)$ (8) and follows at once from

LEMMA 3.2. *Let* U *be an open subset of* B *and* $\omega : U \to A_F^{a,b}$ $(a \geq 0, b \geq 1)$ *a smooth map such that* $\bar\partial\omega(x) = 0$ *for each* $x \in U$. *Then there exists a smooth map* $\tau : U \to A_F^{a,b-1}$ *such that* $\bar\partial\tau(x) = \omega(x)$ *for each* $x \in U$.

Clearly the non triviality of this assertion lies in the smoothness of τ. For the proof we make use of an extension by M. Jurchescu [6] of Cartan's theorems A and B to a certain class of mixed manifolds, which are called in [6] " Cartan manifolds ".

Let us regard the product manifold $X = U \times F$ as a mixed manifold, with U real and F complex. Then the smooth maps $U \to A_F$ are precisely the " complex forms " on X. Moreover F being a Stein manifold entails that X is a Cartan manifold. It follows that the present Lemma is contained in the Corollary 4.2 of [6].

(7) Of course $\mathbf{H}_{\bar\partial}(F, W)$ may be identified with the tensor product $W \otimes \mathbf{H}_{\bar\partial}(F)$.

(8) We recall that $\mathfrak{F}$ is the sheaf on B of germs of smooth sections of $\mathbf{A}_F$ and $\mathfrak{Z}$ is the kernel of $\bar\partial_F : \mathfrak{F} \to \mathfrak{F}$.

Whenever the conditions I) and II) are satisfied, from the exact sequences corresponding to 3.6 (3) and 3.8 (5) of [1] one deduces that there exists a canonical isomorphism between the derived module of $A_B(\mathbf{A}_F(W))$ under $\bar\partial_F$ and the space $A_B(\mathbf{H}_{\bar\partial}(F, W)))$ of smooth forms on B with coefficients in $\mathbf{H}_{\bar\partial}(F, W)$. Then clearly Proposition 2.1 gives at once for the term E_1 of the spectral sequence the result expressed by the Lemma 5.2 of [1] (p. 211).

For the present case, on account of (3.1), we have:

PROPOSITION 3.3. *The isomorphisms* ${}^{(p,q)}k_0^s$ *induce isomorphisms*

$$ {}^{(p,q)}k_1^s : {}^{(p,q)}E_1^s \xrightarrow{\sim} A_B^{s-q,q}(\boldsymbol{\Omega}_F^{p+q-s}(W)) \qquad (p, q, s \geq 0). $$

4. THE TERM E_2

It remains to shown that the Lemma 6.1 of [1] (p. 211) is valid also for the present case.

We need only verify that, in addition to I) and II), the following condition is satisfied:

III) The complex vector bundle $\mathbf{H}_{\bar\partial}(F)$ has some system of holomorphic transition functions, whence it may be viewed as a holomorphic vector bundle on B.

Indeed, besides III) being necessary for the result expressed by the Lemma 6.1 of [1] to make sense (9), it is readily seen, by direct inspection, that actually the proof of that Lemma works for all cases satisfying I), II) and III).

Now, on account of (3.1), III) amounts to the parallel statement for the bundle $\boldsymbol{\Omega}_F(W) = \sum_{a\geq 0} \boldsymbol{\Omega}_F^a(W)$. Then consider the system $\{\psi''_{\alpha\beta} : U_\alpha \cap U_\beta \to \to GL(\Omega_F(\mathbf{C}^n))\}$ of transition functions of $\boldsymbol{\Omega}_F(W)$ obtained by composing the $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Aut}(F)$ with the natural representation of $\mathrm{Aut}(F)$ on $\Omega_F(\mathbf{C}^n)$. Each $\psi''_{\alpha\beta}$ is holomorphic, for, if $\rho : U_\alpha \cap U_\beta \to \Omega_F(\mathbf{C}^n)$ is holomorphic, so too is $\psi''_{\alpha\beta} \circ \rho : U_\alpha \cap U_\beta \to \Omega_F(\mathbf{C}^n)$.

Therefore we have:

PROPOSITION 4.1. *The sum* k_1 *of the isomorphisms* ${}^{(p,q)}k_1^s$ *carries* d_1 *onto* $\bar\partial^{\cdot}$ *As a consequence the* ${}^{(p,q)}k_1^s$ *induce isomorphisms*

$$ {}^{(p,q)}k_2^s : {}^{(p,q)}E_2^s \xrightarrow{\sim} H^{s-q,q}(B, \boldsymbol{\Omega}_F^{p+q-s}(W)) \qquad (p, q, s \geq 0). $$

(9) Because $A_B(H_{\bar\partial}(F, W))$ need be a differential module under the differential $\bar\partial$ acting on base coordinates. We recall that in the case of [1] III) follows from the assumption that every connected component of the structure Lie group G acts trivially on $H_{\bar\partial}(F)$.

5. An application of the spectral sequence

Let us say that a differential form $\omega \in A_E(\hat{W})$ is *partially holomorphic at a point* $z \in E$ is every component ω_i $(i = 1, \cdots, n)$ of ω under a local trivialization of $\hat{W}$ near z, when expressed in terms of local product coordinates (x, y), in a neighbourhood of z is the sum of monomial forms free from the antiholomorphic fibre differentials and with coefficient functions holomorphic in the fibre coordinates. In other words in a neighbourhood of z one has $\omega_i = \Sigma f^{(i)}_{I,J,K}(x, y)\, dx_I \wedge d\bar{x}_J \wedge dy_K$ $(i = 1, \cdots, n)$ and each $f^{(i)}_{I,J,K}(x, y)$ satisfies $\bar{\partial}_y f^{(i)}_{I,J,K}(x, y) = 0$.

One can easily see that this definition is well-posed, *i.e.* does not depend on the choice of the local trivialization of $\hat{W}$ and of the local product coordinates.

We say that ω is a *partially holomorphic form* if it is partially holomorphic near each point of E.

The subspace of $A_E(\hat{W})$ of all partially holomorphic foıms is denoted here $P_E(\hat{W})$. Clearly $P_E(\hat{W})$ is stable under $\bar{\partial}$.

We can prove the following

Proposition 5.1. *The inclusion map* $i : P_E(\hat{W}) \xrightarrow{\subset} A_E(\hat{W})$ *induces an isomorphism on* $\bar{\partial}$*-cohomology.*

Let us denote by $\{E'_r, d'_r\}_{r \geq 0}$ the spectral sequence abutting to $H_{\bar{\partial}}(P_E(\hat{W}))$ associated with the filtration $\{F^s P_E(\hat{W}) = P_E(\hat{W}) \cap F^s A_E(\hat{W})\}_{s \geq 0}$. Then i induces a homomorphism of spectral sequences $\{i_r : E'_r \to E_r\}_{r \geq 0}$. According to a well-known property of the spectral sequences associated with regular filtrations (10), we need only show that i_r is an isomorphism for some r.

By a reasoning similar to the one proving the Lemma 5.1 of [1], it can be seen that there exist canonical isomorphisms

$$^{(p,q)}k'^{s}_0 : {}^{(p,q)}E'^{s}_0 \xrightarrow{\simeq} A^{s-q,q}_B(\Omega^{p+q-s}_F(W)) \qquad (p, q, s \geq 0).$$

Since $d'_0 = 0$ (because plainly $\bar{\partial}F^s P_E(\hat{W}) \subset F^{s+1} P_E(\hat{W})$ for each $s \geq 0$), these induce isomorphisms

$$^{(p,q)}k'^{s}_1 : {}^{(p,q)}E'^{s}_1 \xrightarrow{\simeq} A^{s-q,q}_B(\Omega^{p+q-s}_F(W)) \qquad (p, q, s \geq 0).$$

Then we can apply Proposition 3.3. Since clearly

$$^{(p,q)}k'^{s}_1 = {}^{(p,q)}k^{s}_1 \circ {}^{(p,q)}i^{s}_1 \qquad (p, q, s \geq 0).$$

$i_1 : E'_1 \to E_1$ is an isomorphism.

The above Proposition generalizes the property of a Stein manifold F that the inclusion map $i : \Omega_F \to A_F$ induces an isomorphism on $\bar{\partial}$-cohomology.

(10) See for example [3], p. 80.

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