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De Giorgi's Theorem, for a Class of Strongly Degenerate Elliptic Equations

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Equazioni a derivate parziali. — *De Giorgi's Theorem for a Class of Strongly Degenerate Elliptic Equations* (*). Nota di BRUNO FRANCHI (**) e ERMANNO LANCONELLI (**), presentata (***) dal Socio G. CIMMINO.

RIASSUNTO. — In questa Nota enunciamo, per una classe di equazioni ellittiche del secondo ordine « fortemente degeneri » a coefficienti misurabili, un teorema di hölderianità delle soluzioni deboli che estende il ben noto risultato di De Giorgi e Nash. Tale risultato discende dalle proprietà geometriche di opportune famiglie di sfere associate aglioperatori.

After the fundamental papers of De Giorgi [3] and Nash [15] about the Hölder-continuity of the weak solutions of linear second-order elliptic equations, many authors (see, e.g., [13], [10], [11], [18], [19]) extended this kind of results to more general situations, making suitable hypotheses on the greatest and lowest eigenvalue Λ, λ of the matrix associated to the operator; more precisely, they suppose that Λ/λ is a bounded function and that Λ^{-1}, λ^{-1} fulfill some summability conditions. Unfortunately, these hypotheses are not satisfied in the simple case of the operator $L_\alpha = \partial_x^2 + |x|^{2\alpha} \partial_y^2$, $\alpha > 0$. On the other hand, it is well known that, if α is an integer, the weak solutions of $L_\alpha u = 0$ are smooth functions, since L_α is hypoelliptic [8].

In this paper, even if in a particular situation, we shall obtain regularity results in a strongly degenerate case, via a geometrical approach relying on the geometrical properties of the integral curves of some vector fields associated to the operator.

In what follows, L will be the differential operator $\sum_{i,j=1}^{n} \partial_i (a_{ij}(x) \partial_j) + c(x)$, where c, $a_{ij} = a_{ji}$ are real functions belonging to $L^\infty(R^n)$ and $c \leq 0$. We shall suppose that:

H_1) *there exist* $c_0, c_1 \in R_+$ *such that*:

$$c_0 \sum_{j=1}^{n} \lambda_j^2(x)\, \xi_j^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\, \xi_i\, \xi_j \leq c_1 \sum_{j=1}^{n} \lambda_j^2(x)\, \xi_j^2$$

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$\forall x \in \mathbf{R}^n$, $\forall \xi = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n$, *where* $\lambda_j(x) = \lambda_j^{(1)}(x_1), \cdots, \lambda_j^{(n)}(x_n)$, *and the* $\lambda_j^{(k)}$*'s are real bounded continuous nonnegative functions such that*:

$H_2)$ a) $\lambda_j^{(k)}$ *is a* C^1*-function in* $\mathbf{R} \setminus \{0\}$ *and* $\lambda_j^{(k)}(t) = \lambda_j^{(k)}(-t)$, $j, k = 1, \cdots, n, j \neq k$;

b) λ_j *is Lipschitz-continuous in the* x_j*-variable, uniformly with respect to* x_k, $k \neq j$, $j, k = 1, \cdots, n$;

c) $\exists \rho_{j,k} > 0$ *such that* $0 \leq x_k (\partial_k \lambda_j)(x) \leq \rho_{j,k} \lambda_j(x)$

$$\forall x \in \mathbf{R}^n \qquad \text{if} \quad k, j = 1, \cdots, n, j \neq k.$$

In the sequel, we shall clarify the hypothesis $H_2)$ c); here, we note that it is satisfied in the case of finite order degeneration.

If Ω is a bounded open subset of R^n, we shall denote by $W_\lambda(\Omega)$ ($\mathring{W}_\lambda(\Omega)$) the completion of $C^\infty(\Omega)$ ($C_0^\infty(\Omega)$) with respect to the norm

$$\|u ; W_\lambda(\Omega)\|^2 = \|u ; L^2(\Omega)\| + \sum_{j=1}^{n} \|\lambda_j \partial_j u ; L^2(\Omega)\| .$$

Here and in the following, $\lambda = (\lambda_1, \cdots, \lambda_n)$. Furthermore, we shall say that u belongs to $W_\lambda^{\text{loc}}(\Omega)$ if $\varphi u \in \mathring{W}_\lambda(\Omega)$, $\forall \varphi \in C_0^\infty(\Omega)$.

Finally, we shall denote by $\mathscr{L}$ the following bilinear form on $C^\infty(\Omega) \cap$ $\cap$ $W_\lambda(\Omega)$:

$$\mathscr{L}(u, v) = \int_\Omega \left(\sum_{i,j=1}^{n} a_{i,j} \partial_i u \partial_j v - cuv \right) dx .$$

Obviously, $\mathscr{L}$ can be continued to a bounded bilinear form on $W_\lambda(\Omega)$.

DEFINITION 1. *Let* $f \in L^2_{\text{loc}}(\Omega)$, $u \in W_\lambda^{\text{loc}}(\Omega)$; *we shall say that u is a weak solution of* $Lu = f$ *if* $\mathscr{L}(u, v) = -\int_\Omega fv \, dx$, $\forall v \in C_0^\infty(\Omega)$.

Here, $\mathscr{L}(u, v) = \mathscr{L}(\psi u, v)$, where $\psi \in C_0^\infty(\Omega)$, $\psi v = v$.

DEFINITION 2. *We shall say that the point* $y \in R^n$ *is* λ*-reachable from* $x \in R^n$ *if there exists a broken line connecting x to y, which is a chain of a finite number of integral curves of the vector fields* $\lambda_1 \partial_1, \cdots, \lambda_n \partial_n$.

If Ω *is a subset of* R^n, *we shall say that* Ω *is locally* λ*-connected if* $\forall x \in \Omega$ *and for every neighbourhood* W *of x there exists a neighbourhood* V *of x such that,* $\forall y \in V$, *y is* λ*-reachable from x with a broken line lying in* W.

Now, we have the following extension of De Giorgi's Theorem.

THEOREM 3. *Let* Ω *be an open subset of* R^n *which is locally* λ*-connected; then, every function* $u \in W_\lambda^{\text{loc}}(\Omega)$ *which is a weak solution of* $Lu = 0$ *is locally Hölder-continuous in* Ω.

In fact, the λ-connectedness enables us to define in Ω a suitable pseudo-metric d (see [2], Chapter III), which is " natural " for the operator L. If, in particular, $\lambda_1, \cdots, \lambda_n$ are smooth and nowhere vanishing functions, our pseudo-metric is equivalent to the riemannian metric generated by the quadratic form $\sum_{i=1}^{n} (\lambda_i(x))^{-2} \xi_i^2$. We note that similar operator-shaped metrics can be found in [5] (see also [4]) and in [14] in the case of smooth coefficients. We note also that, in the smooth case, if the Lie algebra generated by $\lambda_1 \partial_1, \cdots, \lambda_n \partial_n$ has constant rank n in Ω, then, by Chow's Theorem (see [7], Chapter 18 and [16], Theorem 7.1), Ω is locally λ-connected.

The properties of the pseudo-metric d enable us to prove our regularity result by a technique which is similar to Moser's [12] one (see also [6]) in the elliptic case. More precisely, we proceed in the following way: our first step is to prove an embedding theorem.

THEOREM 4. *Let Ω_1 be an open subset of R^n which is locally λ-connected and let $\Omega \subseteq \overline{\Omega} \subseteq \Omega_1$ a bounded open subset of Ω_1. Then there exists $\varepsilon_0 \in R_+$ such that $\mathring{W}_\lambda(\Omega)$ is continuously embedded in $\mathring{H}^\varepsilon(\Omega)$ $\forall \varepsilon < \varepsilon_0$, where $\mathring{H}^\varepsilon(\Omega)$ is the usual Sobolev space of order ε.*

Remark a). The number ε_0 can be written explicitly using the constants $\rho_{j,k}$ of the hypothesis H_2); e.g. $\varepsilon_0 = \min\{(1+\rho_{2,1})^{-1}, (1+\rho_{1,2})^{-1}\}$, if $n = 2$.

Remark b). For some particular choice of the functions $\lambda_1, \cdots, \lambda_n$, the preceding theorem partially overlaps with analogous results for weighted Sobolev spaces (see, e.g., [1] and [17] and the references therein).

Remark c). From the local λ-connectedness and the hypothesis H_2), it follows that, for every compact subset K of Ω_1, there exist $\sigma \in]0, 1]$, $C > 0$ such that

$$(*) \qquad d(x, y) \leq C\,|x - y|^\sigma \qquad \forall x, y \in K.$$

Thus if, in particular, the λ_j's are smooth functions, Theorem 4 is contained in [5]. Moreover, we note that, in the smooth case, condition (*) is necessary for the embedding of $\mathring{W}_\lambda(\Omega)$ in $\mathring{H}^\sigma(\Omega)$ (as it is proved in [5]). Now, suppose that, e.g., $n = 2, \lambda_1 = 1, \lambda_2 = b(x_1)$, where b is a smooth nonnegative function. If the estimate (*) holds, then there exists $m \in N \cup \{0\}$ such that $b^{(m)}(0) \neq 0$; this implies Hypothesis H_2) c) is satisfied in a neighbourhood of $(0, 0)$. Thus, the Hypothesis H_2) c) is, in a suitable sense, " necessary " for the embedding theorem.

From Theorem 4 it follows that $\mathring{W}_\lambda(\Omega)$ is (compactly) embedded in $L^q(\Omega)$, for a suitable $q > 2$. Thus, via Moser's iteration method, we can deduce that the weak solutions are locally bounded. Analogously, we can prove that, if $u > 0$ is a weak solution of $Lu = 0$, then $\log u$ is a bounded mean oscillation (BMO) function with respect to the balls $S(x, r)$ of our pseudo-metric. Furthermore,

(Ω, d) is a "homogeneous pseudo-metric space" (see [2], Chapter III); in fact, we have the following result.

THEOREM 5. *Suppose that the hypotheses of Theorem 4 hold. Then, there exists a constant* $A > 0$ *such that*

$$\mu(S(x, r)) \leq A\,\mu(S(x, r/2)),$$

for every ball $S(x, r) \subseteq \Omega$, *where* μ *is the Lebesgue measure in* R^n.

Then, by Theorem 2.2, Chapter III in [2], for the balls $S(x, r)$ a Calderon--Zygmund's decomposition theorem holds; so that we can prove a theorem analogous to John–Nirenberg's [9] one for BMO functions with respect to the balls $S(x, r)$.

From these results, we have the following Harnak's inequality.

THEOREM 6. *Let* Ω *be connected and locally* λ*-connected,* $u \in W_\lambda^{loc}(\Omega)$, $u \geq 0$, u *weak solution of* $Lu = 0$. *Then, for every compact subset* K *of* Ω, *there exists* $C_K > 0$, *which is independent of* u, *such that*

$$\sup_K u \leq C_K \inf_K u.$$

Thus, in the same way as in the elliptic case, we can obtain the Hölder-continuity of the weak solutions.

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