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A note on singular and degenerate abstract equations

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Analisi matematica. — *A note on singular and degenerate abstract equations* (*). Nota di ANGELO FAVINI, presentata (**) dal Socio G. CIMMINO.

RIASSUNTO. — Si considera l'equazione astratta $BA_1 u + A_0 u = h$, dove A_i $(i = 0, 1)$ e B sono convenienti operatori lineari chiusi fra spazi di Banach, A_1 non è necessariamente invertibile, e A_0, A_1 non commutano con B. Si studiano esistenza ed unicità delle soluzioni. Si indicano alcune applicazioni a certe equazioni differenziali degeneri o singolari.

INTRODUCTION

In this note we establish some results of abstract type for operational equations and give some applications of them to certain degenerate or singular differential equations. In fact, we deal with the equation

$$(BA_1 + A_0) u = h , \tag{1}$$

where h is given in the complex Banach space E, u is the solution to be found in the complex Banach space F, B, A_i $(i = 0, 1)$ are suitable linear closed operators from E into itself and from F into E, respectively.

We use the techniques by G. Da Prato and P. Grisvard [2, 4] (see also [3]) to treat (1) in the case in which A_0, A_1 don't commute with B.

Our results permit to consider certain degenerate differential problems in a Banach space-setting. We refer to [1] for a wide literature on the subject, handled mainly by Hilbert space methods.

Proofs and extensions of our results will be given in a forthcoming paper.

ABSTRACT RESULTS

Henceforth we shall assume that A_0, B have everywhere dense domains $D(A_0)$, $D(B)$, respectively, A_0 has a bounded inverse and $D(A_0) \subseteq D(A_1)$. Let $P(\lambda) = \lambda A_1 + A_0$, $\lambda \in \mathbf{C}$. We make the following hypotheses:

H1: $\sigma(B)$, the spectrum of B, is contained in $S_{a,\theta} = \{z \in \mathbf{C} : |\arg z| < \theta, |z| \geq a\}$, $\theta \leq \pi$, $a > 0$; $\sigma(P) = \{z \in \mathbf{C} : P(z)$ has no bounded inverse$\}$, lies outside the sector $\{z \in \mathbf{C} : |\arg z| \leq \theta, |z| > 0\}$. Let $\Gamma_{a,\theta} = \partial S_{a,\theta}$, with the usual orientation.

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H2: For each $\lambda \notin S_{a,\theta}$, we have $\|(B-\lambda)^{-1}\|_{L(E)} \leq C(1+|\lambda|)^{-1}$.

H3: Where $P(\lambda)^{-1}$ exists, we have $\|P(\lambda)^{-1}\|_{L(E;F)} \leq C(1+|\lambda|)^{h}$, $\|A_0 P(\lambda)^{-1}\|_{L(E)} \leq C(1+|\lambda|)^{m}$, where h, m are integers ≥ -1.

H4: For each $\lambda \in \rho(B) \cap \rho(P)$, $\rho(B) = \mathbf{C} \setminus \sigma(B)$, $\rho(P) = \mathbf{C} \setminus \sigma(P)$,

$$\|(B-\lambda)^{-1}[B\,;A_0 P(\lambda)^{-1}]\,x\|_{D(B^k)} \leq C(1+|\lambda|)^{\alpha}\|x\|_E\,,\ x \in D(B)\,,$$

where k is a nonnegative integer, $\alpha \in R$, $C > 0$, and the brackets [;] denote the commutator.

H5: For each $\lambda \in \rho(B) \cap \rho(P)$, we have the estimate

$$\|[B\,;A_0 P(\lambda)^{-1}](B-\lambda)^{-1}\|_{L(E;D(B^k))} \leq C(1+|\lambda|)^{\beta},$$

where k is a nonnegative integer, $\beta \in R$, $C > 0$.

The existence and uniqueness results for (1) are as follows:

THEOREM 1. *Under the assumptions* H1–2–4, *the second inequality in* H3, *if k in* H4 *satisfies $k > \max\{m, \alpha\}$ and the constant* C *there is sufficiently small, then* (1) *has at most one solution.*

THEOREM 2. *Assume* H1–2–3–5. *If k in* H5 *satisfies $k > \max\{h, m, \beta\}$ and the constant* C *there is sufficiently small, then for each $h \in D(B^k)$,* (1) *has at least one solution.*

The proofs of Theorems 1–2 rest on certain complex integrals; the conditions on k ensure their convergence. The solution whose existence is established in Theorem 2 is given by

$$(2\pi i)^{-1}\int_{\Gamma_{a,\theta}} \lambda^{-k} P(\lambda)^{-1}(B-\lambda)^{-1} B^k f\,d\lambda\,,$$

where f is a suitable element in $D(B^k)$. A perturbation argument, (related to the smallness of C), shows that this f exists for given $h \in D(B^k)$. See [2, 4].

Remark 1. Let $E = F = L^p(0, T\,; \mathbf{C}^2)$; $p > 1$, $0 < T < \infty$, $D(B) = W_p^{1,0}(0, T\,; \mathbf{C}^2)$, $(Bu)(t) = u'(t)$. Let A_0, A_1 the operators in E defined by $A_0(t)(x, y) = (tx - y, x)$, $A_1(t)(x, y) = (ty, y)$. It is an easy matter to check that H1–2–3 hold with $h = m = 1$. On the other hand,

$$\|(B-\lambda)^{-1}[B\,;A_0 P(\lambda)^{-1}]\,u\|_{D(B)} \leq C(1+|\lambda|)\|u\|_E$$

and an estimate of the left member in $D(B^k)$, $k > 1$, should imply more regularity for u. This shows that Theorem 1 is necessary. In fact, this is an example of non-uniqueness for (1); a solution exists only if, given $h = (h_1, h_2)$, $h_1(t) = th_2(t)$, and then both $x(t) = h_2(t)$, $y(t) = 0$, and $x(t) = h_2(t) - 1$, $y(t) = t$ furnish solutions for (1).

By use of interpolation spaces we can improve Theorem 2, also as regards the regularity.

THEOREM 3. *Assume* H1–2–3–5 (*with* $k=0$), $p>1$ *and* H6:

$$\| [B ; A_0 P(\lambda)^{-1}] (B-\lambda)^{-1} \|_{L(V;W)} \leq C(1+|\lambda|)^{\gamma},$$

where

$$V=(E ; D(B))_{\theta,p} \quad , \quad W=(E ; D(B^{k+1}))_{\omega,p} \quad , \quad \omega=(k+\theta)(k+1)^{-1},$$
$$0<\theta<1.$$

If $k>\max\{h, m-1, \beta, \gamma\}$ *and* C *in* H6 *is sufficiently small, then* (1) *has at least a solution* u *for all* $h \in W$. *Further*, $A_0 u$, $BA_1 u \in V$.

Notice that Theorem 3 extends [2], where $h=-1$, $m=0$, $\beta, \gamma<0$, $A_1=I$.

Remark 2. H4–6 furnish conditions sufficient for convergence of certain integrals; sometimes it is more suitable to deal with such integrals directly, (see Example 1, below).

APPLICATIONS

We want to give two applications to some differential problems.

Example 1. Let $A_i(t)$, $0 \leq t \leq T < \infty$, $i=0,1$, be linear closed operators from Y into X, where X, Y are complex Banach spaces. Let $p>1$. If $h \in L^p(0, T; X)$, we say that $u=u(\cdot)$ is a strict solution of the problem

$$(2) \qquad \begin{aligned} & d(A_1(t) u(t))/dt + A_0(t) u(t) = h(t), \qquad 0<t \leq T, \\ & \lim_{t \downarrow 0} A_1(t) u(t) = 0, \end{aligned}$$

if $u \in L^p(0, T; Y)$, $t \to A_1(t) u(t)$ is strongly differentiable on $(0, T]$, its derivative belongs to $L^p(0, T; X)$ and (2) holds.

Let $-A(t)$, $0 \leq t \leq T$, be the infinitesimal generator of an analytic bounded semigroup of operators in X, such that $\|(A(t)+\lambda)^{-1}\|_{L(X)} \leq C(1+|\lambda|)^{-1}$, $\operatorname{Re}\lambda \geq 0$, $D(A(t))=D$ and $A(t)v$ is strongly continuously differentiable on $[0, T]$. Define A_0, A_1 by $(A_0 u)(t)=A(t) u(t)$, $u \in L^p(0, T; D)$, $(A_1 u)(t)=tu(t)$, $u \in L^p(0, T; X)=E=F$. Our hypotheses imply the estimate

$$\|(A(t)+\lambda)^{-1}\|_{L(X)} \leq C(1+|\lambda|)^{-1}$$

in a sector independent of t and containing $\operatorname{Re}\lambda \geq 0$. If

$$Sf=(2\pi i)^{-1} \int_{\Gamma_{a,\theta}} P(\lambda)^{-1}(B-\lambda)^{-1} f \, d\lambda, \qquad f \in E,$$

by use of Hardy inequality, we see that Sf, $A_0 Sf \in E$. In order to apply Theorem 3 we need an estimate between intepolation spaces. Now, if $f \in D(B)$, then $B[B; A_0 P(\lambda)^{-1}](B-\lambda)^{-1} f = [B; [B; A_0 P(\lambda)^{-1}]](B-\lambda)^{-1} f + [B; A_0 P(\lambda)^{-1}](B-\lambda)^{-1} Bf$. On the other hand $[B; [B; A_0 P(\lambda)^{-1}]]$ is defined by $\partial^2 (\lambda t A(t)^{-1} + 1)^{-1}/\partial t^2$ and one proves that if $A(t) x$, $x \in D$, is two times continuously strongly differentiable on $[0, T]$, then

$$\int_{\Gamma_{a,\theta}} \lambda^{-1} [B; A_0 P(\lambda)^{-1}](B-\lambda)^{-1} d\lambda$$

defines, by intepolation, a bounded operator from $(E, D(B))_{\theta,p}$ into itself, with a small norm (a simple change-of-variable argument) and thus Theorem 3 applies.

Example 2. Consider the problem

$$\partial(m(t,x)\, u(t,x))/\partial t + A(t,x,D)\, u(t,x) = h(t,x), \qquad 0 < t \leq T, \quad x \in \Omega,$$

$$\lim_{t \downarrow 0} m(t,x)\, u(t,x) = 0, \qquad x \in \Omega'$$

where Ω is a bounded open domain in R^n with a smooth boundary, $m(t,x) \geq 0$ is a continuous function on $[0,T] x\overline{\Omega}$ with a derivative $\partial m(t,x)/\partial t$ such that $m(t,x) > 0$ on $[0,T] x\Omega$, $|\partial m(t,x)/\partial t| \leq Cm(t,x)$, and there is a function $\alpha(x)$, continuous on $\overline{\Omega}$, for which $C_1 \alpha(x) \leq m(t,x) \leq C_2 \alpha(x)$, for certain $C_1, C_2 > 0$. $A(t,x,D)$ is a suitable differential operator. First of all, we need some spaces " ad hoc ".

If β is a positive measurable function on Ω, then $L^2_\beta(\Omega)$ denotes the set of all complex-valued measurable functions on Ω such that $\beta u \in L^2(\Omega)$. Let $L^2(0,T; L^2_{\sqrt{\alpha}}(\Omega)) = V$, $L^2(0,T; L^2_{1/\sqrt{\alpha}}(\Omega)) = V'$, $L^2(0,T; L^2(\Omega)) = H$.

We assume that $A(t,x,D)$ defines an invertible operator $A_0(t)$, $0 \leq t \leq T$, with the common domain $H^{2m}(\Omega) \cap H^m_0(\Omega)$, e.g., and range $L^2(\Omega)$, in such a way that $\|A_0(t) A_0(s)^{-1}\|_{L(L^2_{1/\sqrt{\alpha}}(\Omega))} \leq M$, $0 \leq t, s \leq T$. Further, $A_0(t)$ defines an operator A_0 from $L^2(0,T; D)$ into V', where $D = \{u \in H^{2m}(\Omega) \cap H^m_0(\Omega)$: $A_0(t) u \in L^2_{1/\sqrt{\alpha}}(\Omega)$ for all $t \in [0,T]\}$ and, at last,

$$\operatorname{Re} \int_0^T \int_\Omega A(t,x,D)\, v(t,x)\, \overline{v}(t,x)\, dt\, dx \geq C(\|v\|^2_{L^2(0,T;H^m(\Omega))} + \|v\|^2_H).$$

Notice that the operator A_1 defined by $m(t,x)$ is an isomorphism from V onto V'. From these assumptions we easily deduce

$$\|P(\lambda)^{-1} f\|_V \leq C(1+|\lambda|)^{-1} \|f\|_{V'}, \quad \|A_0 P(\lambda)^{-1} f\|_{V'} \leq C \|f\|_{V'},$$

and thus H2–3 are satisfied.

In order to estimate the norm of $[B ; A_0 P(\lambda)^{-1}]$ in the spaces V' and $(V', D(B))_{\theta,p}$ we need further assumptions. They turn out from the formal identity $\partial (A_0(t)(\lambda A_1(t) + A_0(t))^{-1})/\partial t = -\lambda(\lambda T(t)+1)^{-1} A_1'(t) A_0(t)^{-1} \cdot (\lambda T(t)+1)^{-1} + \lambda(\lambda T(t)+1)^{-1} A_1(t) A_0(t)^{-1} A_0'(t) A_0(t)^{-1} (\lambda T(t)+1)^{-1}$, with $T(t) = A_1(t) A_0(t)^{-1}$, $A_1'(t) A_0(t)^{-1} (\lambda T(t)+1)^{-1} = A_1'(t)(\lambda A_1(t) + A_0(t))^{-1}$.

We then see that if $A_0(t)$, $A_1(t)$ are sufficiently smooth, $A_1'(t)$ maps $L^2_{\sqrt{\alpha}}(\Omega)$ into $L^2_{1/\sqrt{\alpha}}(\Omega)$ with a uniform bound, $\| A_0'(t) A_0(t)^{-1} \|_{L(L^2_{1/\sqrt{\alpha}}(\Omega))} \leq C$, then $\| [B ; A_0 P(\lambda)^{-1} \|_{L(V')} \leq C(1 + |\lambda|)^{-1}$.

A corresponding estimate is obtaine in the space $(V', D(B))_{\theta,p}$ if we repeat the argument as in Example 1, that is, by use of the interpolation property.

The case $p \neq 2$ can be considered analogously. Notice that no estimate in interpolation spaces is necessary if we are content with a weaker type of solution, the so-called " strong solution " according [2]. This shall be desplayed in the forthcoming paper.

LITERATURE

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