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A method for treating a class of nonlinear diffusion problems

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Analisi matematica. — *A method for treating a class of nonlinear diffusion problems.* Nota di STAVROS BUSENBERG e MIMMO IANNELLI, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si presenta un metodo di soluzione di una classe di problemi di diffusione nonlineare che hanno origine dalla teoria delle popolazioni con struttura di età.

1. INTRODUCTION

In this note we present a method for studying the existence, the uniqueness and asymptotic behavior of the solutions of a class of nonlinear diffusion problems. These problems are motivated by models of age dependent population dynamics which find their origin in the article [4] by Gurtin and MacCamy. The problem can be stated as follows for $(a, t, x) \in [0, \infty) \times [0, T] \times J$, J an open, bounded interval in $\mathbf{R}$.

$$(P) \quad \begin{cases} u_a + u_t + \mu(a, t, x)\, u = b(t, x, P, P_x)\, u_x + c(t, x, P, P_x, P_{xx})\, u, \\ u(0, t, x) = \displaystyle\int_0^\infty \beta(a, t, x)\, u(a, t, x)\, da, \\ u(a, 0, x) = u_0(a, x), \\ \rho u + \nu u_x = 0 \quad \text{on } \partial J, \; u \geq 0 \quad \text{on } \bar{J}, \; u(a, t, x) \to 0 \quad \text{as } a \to \infty. \end{cases}$$

Here, $P(t, x) = \displaystyle\int_0^\infty u(a, t, x)\, da$, and ρ and ν are constants. In our notation, subscripts denote partial differentiation with respect to the subscripted variable, and the independent variables are suppressed when this does not lead to ambiguity. Thus, in this notation, $u_t = (\partial u/\partial t)(a, t, x)$. Specific hypothese on b, c, μ and β will be given later. A specific case included in our formulation is:

$$(1.1) \qquad bu_x + cu = [uk(P)\, P_x]_x .$$

(*) Nella seduta del 13 marzo 1982.

When $k(\mathrm{P}) = 1$, this reduces to a problem studied by MacCamy [7] with the special choice of $\mu =$ constant, and $\beta = \bar{\beta} e^{-\alpha a}$; α and $\bar{\beta}$ constants. The case $k(\mathrm{P}) = 1/\mathrm{P}$ is the age dependent generalization of a diffusion mechanism studied by Busenberg and Travis [1].

Our formulation does not include the case where the term u_{xx} is present in the diffusion mechanism. This type of diffusion in nonlinear age dependent models has been studied by Di Blasio [2, 3], and Webb [8], who treat a linear diffusion operator; and by Langlais [6] who studies the nonlinear operator $[\mathrm{P}u_x]_x$. Gurtin and Mac Camy [5] provide a discussion of nonlinear diffusion models for age-structured populations.

2. Formal reduction of the problem

We proceed to perform a formal transformation of a special case of (P) where β and μ depend continuously on a only, and which we denote as the problem (P_a). It turns out that the analysis of (P) is based on the treatment of (P_a). We place the following restrictions on β and μ:

There exists $\gamma > 0$ such that $\delta(a) = \gamma + \beta(a) - \mu(a)$, satisfies $0 \leq \delta(a) \leq 2\gamma$. Moreover $\beta(a) \geq 0$ $\mu(a) \geq 0$.

Let $\phi(t, t_0, x)$ denote the solution of the problem

$$(2.1) \qquad \begin{cases} \phi_t(t, t_0, x) = -b(t, x, \mathrm{P}(t, \phi(t, t_0, x)), \mathrm{P}_x(t, \phi(t, t_0, x))), \\ \phi(t_0, t_0, x) = x, \end{cases}$$

and define

$$(2.3) \qquad w(a, t, x) = u(a, t, \phi(t, 0, x))/\mathrm{P}(t, \phi(t, 0, x)).$$

Then, the triplet (w, P, ϕ) satisfies the following problems:

$$(P_1) \qquad \begin{cases} w_a + w_t + \mu(a) w = \left[\gamma - \int_0^\infty \delta(a) w(a, t, x) \, \mathrm{d}a\right] w, \\ w(0, t, x) = \int_0^\infty \beta(a) w(a, t, x) \, \mathrm{d}a, \\ w(a, 0, x) = w_0(a, x) = u_0(a, x)/\mathrm{P}_0(x), \\ \int_0^\infty w(a, t, x) \, \mathrm{d}a = 1 \quad , \quad \nu w_{0x} = 0 \quad \text{on } \partial \mathrm{J}, w(a, t, x) \to 0 \\ \qquad \text{as } a \to \infty \quad , \quad w \geq 0. \end{cases}$$

$$
(P_2) \quad \begin{cases} P_t = bP_x + cP - \gamma P + P \displaystyle\int_0^\infty \delta(a)\, w(a, t, \phi(0, t, x))\, da, \\ P(0) = P_0 = \displaystyle\int_0^\infty u_0(a)\, da, \\ P \geq 0 \quad , \quad \rho P + \nu P_x = 0 \qquad \text{on} \quad \partial J, \end{cases}
$$

$$
(P_3) \quad \begin{cases} \phi_t = -b(t, x, P(t, \phi), P_x(t, \phi)), \\ \phi(t_0, t_0, x) = x. \end{cases}
$$

The problem (P_1) is seen to be independent of P and ϕ, and the variable x occurs there only as a parameter. Note that, when $b = 0$, the problem (P_3) is trivial with solution $\phi(t, t_0, x) = x$. Here, the reduced problem consists of (P_1) and (P_2) only, and the analysis is simpler.

The advantage of this particular reformulation of the problem (P_a) is due to the separation of the diffusion from the age dependent dynamics. The major difficulties in treating (P_2) will come from the nonlinearities which must be specified before the problem can be analyzed. Each specific type of nonlinear diffusion term may require its own special approach. We shall see, however, that what is needed is an existence theorem and some regularity conditions on the solutions of problems (P_2)–(P_3).

3. ANALYSIS OF (P_1)

We start by studying problem (P_1) whose solution enters in the definition of (P_2) and of (P_3). We define the following sets

$$
E = [0, T] \times \bar{J} \quad ; \quad F = [0, \infty) \times [0, T] \quad ; \quad G = [0, \infty) \times \bar{J};
$$
$$
H = [0, \infty) \times [0, T] \times \bar{J}
$$

and we denote their respective elements by

$$
(t, x) \quad ; \quad (a, t) \quad ; \quad (a, x); \qquad \text{and} \quad (a, t, x).
$$

Now, letting

$$
R = \{w \mid w \in C(E; L^1(0, \infty)), w_x \in C(E, L^1(0, \infty)),
$$
$$
w_{xx} \in C(E, L^1(0, \infty)), w \geq 0\},
$$

we can give the following result.

THEOREM 3.1. *Let* $w_0 \in C^1([0,\infty)\,;\,C(\bar{J})) \cap C([0,\infty)\,;\,C^2(\bar{J})) \cap L^1((0,\infty), C^2(\bar{J}))$, *be such that*

$$w_0(a,x) \geq 0\,, \int_0^\infty w_0(a,x)\,da = 1, \qquad \textit{and for } x \in \bar{J} \quad w_0(a,x) \to 0 \text{ as } a \to \infty, \tag{3.1}$$

$$w_0(0,x) = \int_0^\infty \beta(a)\,w_0(a,x)\,da\,, \tag{3.2}$$

$$\nu w_{0x} = 0 \quad \textit{on } \partial J\,, \tag{3.3}$$

$$w_{0a} \in L^1((0,\infty)\,;\,C(\bar{J})) \quad \textit{and} \quad w_{0a}(0,x) + \mu(0)\,w_0(0,x) = \int_0^\infty \beta(a)\,[w_{0a}(a,x) + \mu(a)\,w_0(a,x)]\,da\,. \tag{3.4}$$

Then, there exists a unique solution $w \in R \cap C^1(F\,;\,C(\bar{J})) \cap C(F\,;\,C^2(\bar{J})) \cap L^1((0,\infty), C^2(\bar{J}))$ *of the problem* (P_1), *such that,* $\nu w_{0x} = 0$ *on* ∂J.

The conditions (3,2) and (3.4) are simply a requirement that the initial data agree with the dynamic equations of (P_1) at $t = 0$. These conditions are imposed because we have chosen to seek " strict " solutions of the problem (P) and need to avoid singularities along the characteristic $t = a$. It is for the same reason that we have imposed smoothness restrictions on the dependence of the initial data on the space variable x. It can be shown that less regular data lead to a similar result for less regular solutions.

4. JUSTIFICATIONS OF THE FORMAL REDUCTION

We will now give a general lemma for the problem (P_a) which, as we will show in the next section, can be used to treat specific forms of nonlinear diffusion operators. The purpose of this result is to show that the formal reduction performed in section 2 leads to a method for resolving the original problem.

We seek a strict solution in the sense that

$$u \in C(F\,;\,C^2(\bar{J})) \cap C^1(F\,;\,C(\bar{J})) \cap L^1((0,\infty), C^2(\bar{J}))\,. \tag{4.1}$$

and

$$\left\{\begin{array}{l}\text{the functions } (t,x) \to b(t,x,P(t,x),P_x(t,x)) \text{ and } (t,x) \to \\ \to c(t,x,P(t,x),P_x(t,x),P_{xx}(t,x)) \text{ are continuous on E.}\end{array}\right. \tag{4.2}$$

We will use the following conditions on u_0:

(4.3) $\quad u_0 \geq 0 \quad , \quad u_0 \in L^1((0,\infty), C^2(\bar{J}))$,

(4.4) $$P_0(x) = \int_0^\infty u_0(a,x)\,da > 0 \qquad \text{for} \quad x \in J,\ \rho P_0 + \nu P_{0x} = 0 \qquad \text{on } \partial J,$$

and

$$b(0,x,P_0(x),P_{0x}(x)) \quad , \quad c(0,x,P_0(x),P_{0x}(x),P_{0xx}(x))$$

are continuous on $\bar{J}$.

(4.5) $\quad w_0(a,x) = u_0(a,x)/P_0(x)$ is defined in $\bar{J}$ and satisfies the conditions imposed in Theorem 3.1.

We can now give our general Lemma.

LEMMA 4.1. *Suppose that* (u_0, P_0, w_0) *satisfies* (4.3)-(4.5), *and for any* $f, \lambda \in C^1(E)$, *both non negative, and satisfying the same boundary conditions as* P_0 *on* ∂J, *the problem* $(P_{2\lambda})$-(P_3):

$$(P_{2\lambda}) \quad \begin{cases} P_t = b(t,x,P,P_x)P_x + c(t,x,P,P_{xx})P - \gamma P + \lambda(t,\phi(0,t,x))P + f, \\ P(0,x) = P_0(x),\ \rho P + \nu P_x = 0 \quad \text{on } \partial J,\ P > 0 \quad \text{for } x \quad \text{in } J, \end{cases}$$

with (P_3) *defined as in section 2, has a unique solution*

$$(P,\phi) \in (C^1([0,T];C(\bar{J})) \cap C([0,T];C^2(\bar{J})),$$
$$C^1([0,T]\times[0,T];C(\bar{J})) \cap C([0,T]\times[0,T];C^2(\bar{J}))),$$

such that (4.2) *holds.*

Then the problem (P_a) *has a unique strict solution given by*

$$u(a,t,x) = P(t,x)\,w(a,t,\phi(0,t,x)), \tag{4.6}$$

where w *is the unique strict solution of the problem* (P_1).

We now turn to specific examples that illustrate the applicability of this method of treating the problem.

5. EXISTENCE AND UNIQUENESS RESULTS

We consider two cases for which all the steps of our method can be carried out. The first case leads to a problem with $b = 0$, hence, one where the problem (P_3) is trivial. The second case will require the consideration of both (P_2) and (P_3).

For the first example, we take

$$bu_x + cu = k \frac{u}{P} P_{xx}, \tag{5.1}$$

and for the second,

$$bu_x + cu = k \frac{P_x}{P} u_x + k \left[\frac{P_{xx}}{P} - \frac{P_x^2}{P^2} \right] u = k \left(\frac{u}{P} P_x \right)_x. \tag{5.2}$$

We will consider Newmann boundary conditions: $\rho = 0$ $\nu \neq 0$; appropriate mixed conditions can also be treated. Because of the form of the diffusion terms in (5.1) and (5.2) we need to place the condition $P > 0$ on $\bar{J}$.
Note that (5.1) is a small gradient version of (5.2) in the sense that, when the quadratic terms in u_x and P_x are dropped from (5.2), we obtain (5.1).

The problem $(P_{2\lambda})$ for (5.1) is

$$\begin{cases} P_t + kP_{xx} - \gamma P + \lambda(t, x) P + f \quad , & P(0, x) = P_0(x), \\ P > 0 \quad \text{in } \bar{J} \quad , \quad P_x = 0 & \text{on } \partial J, \end{cases} \tag{5.3}$$

with

$$\lambda(t, x) = \int_0^\infty \delta(a)\, w(a, t, x)\, da. \tag{5.4}$$

From classical results, it can be seen that the conditions of Lemma 4.1 are satisfied, and we immediately have

THEOREM 5.1. *Let* u_0 *satisfy* (4.3)–(4.5), *and let* $P_0(x) > 0$ *on* $\bar{J}$. *Then the problem* (P) *with condition* (5.1) *has a unique strict solution.*

In the case of (5.2), the problems $(P_{2\lambda})$–(P_3) become

$$\begin{cases} P_t = kP_{xx} - \gamma P + \lambda(t, \phi(0, t, x)) + f \quad , & P(0, x) = P_0(x), \\ P > 0 \quad \text{on } \bar{J} \quad , \quad P_x = 0 & \text{on } \partial J, \end{cases} \tag{5.5}$$

with λ still given by (5.4), and

$$\begin{cases} \phi_t(t, t_0, x) = -P_x(t, \phi(t, t_0, x))/P(t, \phi(t, t_0, x)), \\ \phi(t_0, t_0, x) = x. \end{cases} \tag{5.6}$$

We now have the following result.

THEOREM 5.2. *Let* u_0 *satisfy conditions* (4.3)–(4.5) *and let* $P_0(x) > 0$ *on* $\bar{J}$. *Then the problem* (*P*) *with condition* (5.2) *has a unique strict solution.*

The proof of this result is based on setting up a fixed point problem

$$\tau : C^1([0, T]; C^1(\bar{J})) \to C^1([0, T]; C^1(\bar{J})),$$

with τ defined as follows. Pick $\eta \in C^1([0, T]; C^1(\bar{J}))$, construct $P(t, x \mid \eta)$ (here, we explicitely show the dependence on η) by solving the problem (5.5) with $\eta(t, x)$ replacing $\lambda(t, \phi(0, t, x))$ in (5.5). Using this P in (5.6) $\phi(t, t_0, x \mid \eta)$ is found.

Finally, set

$$(\tau\eta)(t, x) = \lambda(t, \phi(0, t, x \mid \eta)).$$

Appropriate estimates show that this construction of τ is possible and that τ has a unique fixed point ξ. Finally, $(P(t, x \mid \xi), \phi(t, t_0, x \mid \xi))$ is a solution of (5.5)–(5.6) and the conditions of Lemma 4.1. are satisfied.

6. Conclusions and extensions

We have presented the essentials of a general approach for treating age-dependent nonlinear diffusion problems. This approach allows the handling of birth and death rates, β and μ, that can have a general dependence on the age variable a. We have illustrated the validity of our approach by employing it to resolve two specific problems of this type. This method can also be applied to the analysis of nonlinear diffusion terms of the form $(uP_x)_x$ which have been studied only with special conditions on β and μ [7]. It can also be applied to the treatment of the problem with β and μ depending on t and x. The details of the proofs and of the above generalizations will be presented elsewhere.

We finally note that our approach leads to a way for determining the asymptotic behavior of the solutions of (P) when $t \to \infty$. In fact, the asymptotic behavior of w can be obtained directly from (P_1), and from this, the behavior of $u(a, t, \phi(t, 0, x))/P(t, \phi(t, 0, x))$ is determined.

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