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Subproducts defined by means of subdirect products

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Algebra. — *Subproducts defined by means of subdirect products.*
Nota di FRANS LOONSTRA, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si supponga che l'anello R ammetta una decomposizione come prodotto subdiretto $R = \tilde{\times}_{\alpha \in A} R_\alpha$ di anelli $R_\alpha \neq 0$, tali che per $S_\alpha = R \cap R_\alpha$ si abbia $\mathrm{Ann}_{R_\alpha} S_\alpha = 0$ ($\forall \alpha \in A$), e sia $S = \bigoplus_{\alpha \in A} S_\alpha$. Si scelga un R-modulo (destro) M che sia libero da torsione rispetto ad S, cioè $\mathrm{Ann}_M S = 0$; allora M può essere rappresentato come prodotto subdiretto irridondante $M \cong \tilde{\times}_{\alpha \in A} M_\alpha$ degli R_α-moduli M_α liberi da torsione rispetto ad S_α. Si fa uno studio di un subprodotto generale di una classe **C** di R-moduli $M^{(i)}$ ($i \in I$), dove C è determinato per mezzo di epimorfismi e relazioni.

1. INTRODUCTION

We assume that the (associative) ring R (with $1_R = 1$) admits a decomposition as a subdirect product

(1) $$R = \tilde{\times}_{\alpha \in A} R_\alpha$$

of rings $R_\alpha \neq 0$ $(\alpha \in A)$ such that $S_\alpha = R \cap R_\alpha$ satisfies the condition

(2) $$\mathrm{Ann}_{R_\alpha} S_\alpha = \{ r_\alpha \in R_\alpha \mid r_\alpha S_\alpha = 0 \} = 0 \qquad (\forall a \in A).$$

In particular, $S_\alpha \neq 0$ ($\forall a \in A$), i.e. the subdirect representation (1) of R is *irredundant* in the sense that none of R_α can be omitted from (1). Setting $S = \bigoplus_{\alpha \in A} S_\alpha$ we have

(3) $$\mathrm{Ann}_R S = 0 .$$

Since S_α is an ideal of R_α (and of R), S is an ideal of R. R_α is even a rational extension of S_α (both viewed as right R_α-modules (notation: $S_\alpha \subseteq_r R$) and for a similar reason we have $S \subseteq_r R$. This implies that R_α is an essential extension of the right R_α-module S_α (notation: $S_\alpha \subseteq_e R_\alpha$) and $S \subseteq_e R$.

Denoting the canonical projection $R \to R_\alpha$ by π_α, $\mathrm{Ker}\, \pi_\alpha = P_\alpha$, we conclude that $P_\alpha = \mathrm{Ann}_R S_\alpha$. Let M be a (right) R-module which is S-*torsionfree*

(*) Nella seduta del 13 marzo 1982.

in the sense that

$$\text{(4)} \qquad \mathrm{Ann_M\, S} = \{m \in \mathrm{M} \mid ms = 0 \quad \text{for all} \quad s \in \mathrm{S}\} = 0 .$$

We wish to have a representation of M as an irredundant subdirect product of R_α-modules M_α. To this end, we define

$$\text{(5)} \qquad N_\alpha = \mathrm{Ann_M}\, S_\alpha \qquad (\alpha \in A) ,$$

and we observe that

$$\bigcap_\alpha N = \bigcap_\alpha \mathrm{Ann_M}\, S_\alpha = \mathrm{Ann_M}\Big(\sum_\alpha S_\alpha\Big) = \mathrm{Ann_M}\, S = 0 .$$

If we set $M_\alpha = M/N_\alpha$, then we obtain a representation of M as a subdirect product of R-modules:

$$\text{(6)} \qquad M \cong \underset{\alpha \in A}{\tilde{\times}} M_\alpha .$$

In case M = R, (6) specializes to (1). One can prove the following statements:

(*i*) M_α is in a natural way an R_α-module; indeed $M_\alpha (\mathrm{Ker}\, \pi_\alpha) = M_\alpha P_\alpha = 0$.

(*ii*) M_α is S_α–torsionfree; for if $m_\alpha S_\alpha = 0$, and $m \in M$ has m_α as α-coordinate, then $m\, S_\alpha = 0$, i.e. $m \in N_\alpha$ and $m_\alpha = 0$.

Then one can prove the following theorem (see: Fuchs–Loonstra [1]):

1.1. *Let* R *be a ring as above. If* M *is an* S-*torsionfree* R-*module, then the non-zero* M_α*'s in* (6) *yield an irredundant representation of* M *as a subdirect product of* S_α-*torsionfree* R_α-*modules* M_α.

2. Subproducts and their decomposition

Among the submodules of a direct product the subdirect products play an important role. However—in general—not much is known about their structure. In the case of a subdirect product M of two R-modules M_1, M_2 we know that there exists an R-module F and two R-epimorphism α_1, α_2; $\alpha_1 : M_1 \to F$, $\alpha_2 : M_2 \to F$, such that M can be represented as $M = \{(m_1, m_2) \mid \alpha_1 m_1 = \alpha_2 m_2\}$.

In general, a subdirect product $M = \underset{\alpha}{\tilde{\times}} M$ of more than two R-modules M_α is not such a *special subdirect product*, i.e. there does not always exist a module F and epimorphisms $\phi_\alpha : M_\alpha \to F\ (\alpha \in A)$ such that M is the R-module of elements $m = (\cdots, m_\alpha, \cdots, m_\beta, \cdots) \in \prod_{\alpha \in A} M_\alpha$ with the property

$$\cdots = \phi_\alpha m_\alpha = \cdots = \phi_\beta m_\beta = \cdots .$$

For more than two modules—in general—no satisfactory description of subdirect products is even available. In the *finite* case, however, we know more of the submodules of a direct sum $\overset{k}{\underset{i=1}{\oplus}} M_i$. If M is *any* submodule of $M^* = \overset{k}{\underset{i=1}{\oplus}} M_i$ (M not necessarily a subdirect sum), then we have, for each $i = 1, \cdots, k$, a homomorphism

$$\alpha_i : M_i \to F = M^*/M \; (\alpha_i m_i = m_i + M),$$

such that $(m_1, m_2, \cdots, m_k) \in M^*$ belongs to M exactly if

$$\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_k m_k = 0 .$$

This idea will be generalized in the following. Therefore we start

(*i*) with a ring R admitting a decomposition (1) as subdirect product $R = \underset{\alpha \in A}{\tilde{\times}} R_\alpha$ with the properties (2): $\mathrm{Ann}_{R_\alpha} S_\alpha = 0$ $(\forall \alpha \in A)$ and

(*ii*) with a right R-module M which is S-torsionfree.

We have seen (see 1.1) that M can be represented as a subdirect product of R_α-modules M_α, where $M_\alpha = M/N_\alpha$, $N_\alpha = \mathrm{Ann}_M S_\alpha$ $(\alpha \in A)$, $\mathrm{Ann}_{M_\alpha} S_\alpha = 0$ $(\alpha \in A)$. It may happen that some of the M_α in the decomposition of M are zero; therefore we omit the irredundancy of the decomposition.

Suppose that M and F are both S-torsionfree R-modules (R as above) and $\phi : M \to F$ an R-epimorphism. Then using the decompositions $M = \underset{\alpha}{\tilde{\times}} M_\alpha$, $F = \underset{\alpha}{\tilde{\times}} F_\alpha$ we prove that ϕ induces—for each pair (M_α, F_α) an R—epimorphism $\phi_\alpha : M_\alpha \to F_\alpha$ $(\alpha \in A)$. Indeed, $M_\alpha = M/N_\alpha$, $N_\alpha = \mathrm{Ann}_M S_\alpha$, $F_\alpha = F/K_\alpha$, $K_\alpha = \mathrm{Ann}_F S_\alpha$. Then $N_\alpha = \{m \in M \mid mS_\alpha = 0\}$, $K_\alpha = \{f \in F \mid f S_\alpha = 0\}$.

$$\begin{array}{ccc} M = \cdots \tilde{\times} & M_\alpha & \tilde{\times} \cdots \\ \phi \downarrow & & \downarrow \phi_\alpha \\ F = \cdots \tilde{\times} & F_\alpha & \tilde{\times} \cdots \end{array}$$

If $m \in N_\alpha$ then $\phi(m) \in K_\alpha$, since $\phi(m) S_\alpha = \phi(mS_\alpha) = 0$. This mean s that $\phi(N_\alpha) \subseteq K_\alpha$, and since ϕ is an epimorphism, ϕ induces an epimorphism $\phi_\alpha : M_\alpha \to F_\alpha$, defined by $\phi_\alpha(m + N_\alpha) = \phi(m) + K_\alpha$, or $\phi_\alpha(m_\alpha) = \phi(m) + K_\alpha = f_\alpha \in F_\alpha$.

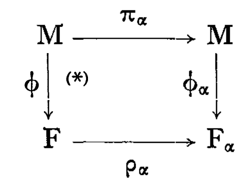
The epimorphism ϕ_α is an R-epimorphism of the R-module M_α onto the R-module F_α; we may even consider ϕ_α as an R_α-epimorphism of the R_α-module M_α onto the R_α-module F_α. Indeed: if $m_\alpha r_\alpha = m_\alpha r$, and, in a similar way, $f_\alpha r_\alpha = f_\alpha r$, then we have:

$$\phi_\alpha(m_\alpha r_\alpha) = \phi_\alpha(m_\alpha r) = \phi_\alpha(m_\alpha) r = f_\alpha r = f_\alpha r_\alpha .$$

That implies

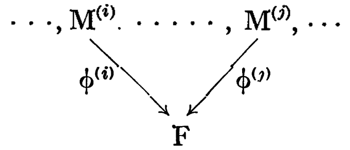
(i) the R-epimorphism $\phi : M \twoheadrightarrow F$ induces (uniquely) R_α-epimorphisms $\phi_\alpha : M_\alpha \twoheadrightarrow F_\alpha$ $(\alpha \in A)$, and

(ii) the diagram $(*)$, where $\rho_\alpha : F \twoheadrightarrow F_\alpha$ is the canonical projection $\rho_\alpha : F \twoheadrightarrow F/K_\alpha = F_\alpha$ is commutative.



Suppose that we have the R-modules $M^{(i)}$ $(i \in I)$, F and R-epimorphisms $\phi^{(i)} : M^{(i)} \twoheadrightarrow F$ $(i \in I)$; R is again as above and the modules $M^{(i)}$ $(i \in I)$ and F are S-torsionfree.

(7)



We consider the R-module $M \subseteq M^* = \prod_i M^{(i)}$, consisting of those elements $m = (m^{(i)}) \in M^*$, satisfying the relations

$$
(8) \qquad \begin{cases} \phi^{(i_1)}(m^{(i_1)}) + \phi^{(i_2)}(m^{(i_2)}) + \cdots \phi^{(i_l)}(m^{(i_l)}) = 0\,, \\ \phi^{(i'_1)}(m^{(i'_1)}) + \cdots \qquad\qquad = 0\,, \\ \cdots\cdots\cdots\cdots\cdots \qquad\qquad = 0\,, \end{cases}
$$

each relation of (8) consisting of finitely many terms. M is an R-submodule of M^* and is called a *general subproduct* of the $\{M^{(i)}\}$, determined by the $\{\phi^{(i)}\}$ and the relations (8) and denoted by

$$M = \{M^{(i)} \,;\, \phi^{(i)} \,;\, F \mid i \in I\} \text{ (1)}.$$

Under the conditions that all $M^{(i)}$ and F are S-torsionfree, M is also S-torsionfree.

Indeed, we have $\mathrm{Ann}_{M^{(i)}} S = 0$ $(i \in I)$ and $m\,S = (\cdots, m^{(i)}, \cdots)\,S = 0$ implies $m^{(i)}\,S = 0$, and that means that all $m^{(i)} = 0$, since the $M^{(i)}$ are S-torsionfree. But then also M is S-torsionfree.

For each $M^{(i)}$ $(i \in I)$ we have a decomposition as a subdirect product

$$M^{(i)} = \underset{\alpha \in A}{\tilde{\times}} M^{(i)}_\alpha \qquad (i \in I)\,,$$

where the $M^{(i)}_\alpha$ are also R_α-modules and S_α-torsionfree.

(1) See e.g. L. Fuchs–F. Loonstra [2] and F. Loonstra [3], [4].

Using the results (*i*) and (*ii*) (of p. 4) we see that the diagram (7) induces (for each $a \in \mathrm{A}$) a diagram (7*a*)

(7*a*)
$$\begin{array}{ccc} \cdots, \mathrm{M}_\alpha^{(i)}, \cdots\cdots, & & \mathrm{M}_\alpha^{(j)}, \cdots \\ \phi_\alpha^{(i)} \searrow & & \swarrow \phi_\alpha^{(j)} \\ & \mathrm{F}_\alpha & \end{array}$$

such that the corresponding R-homomorphisms $\phi_\alpha^{(i)} : \mathrm{M}_\alpha^{(i)} \to \mathrm{F}_\alpha$ $(i \in \mathrm{I})$ can be considered as R_α-epimorphisms, where

$$m_\alpha^{(i)}\, r_\alpha = m_\alpha^{(i)}\, r, \quad \text{if} \quad r_\alpha = \pi_\alpha r,$$

and

$$\phi_\alpha^{(i)}\,(m_\alpha^{(i)}\, r_\alpha) = \phi_\alpha^{(i)}\,(m_\alpha^{(i)})\, r = f_\alpha^{(i)}\, r = f_\alpha^{(i)}\, r_\alpha\,.$$

The epimorphisms $\phi^{(i)}$ $(i \in \mathrm{I})$ and the relations (8) determine (for each $\alpha \in \mathrm{A}$) uniquely the epimorphisms $\phi_\alpha^{(i)}$ and the relations

(8*a*)
$$\begin{cases} \phi_\alpha^{(i_1)}\,(m_\alpha^{(i_1)}) + \phi_\alpha^{(i_2)}\,(m_\alpha^{(i_2)}) + \cdots + \phi_\alpha^{(i_l)}\,(m_\alpha^{(i_l)}) = 0\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots \qquad = 0\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots \qquad = 0\,. \end{cases}$$

Indeed, we have $\phi_\alpha^{(i_1)}\,(m_\alpha^{(i_1)}) = \phi^{(i_1)}\,(m^{(i_1)}) + \mathrm{K}_\alpha$, and addition gives

$$\phi_\alpha^{(i_1)}(m_\alpha^{(i_1)}) + \phi_\alpha^{(i_2)}\,(m_\alpha^{(i_2)}) + \cdots + \phi_\alpha^{(i_l)}\,(m_\alpha^{(i_l)}) = 0 + \mathrm{K}_\alpha = 0 \in \mathrm{F}_\alpha\,, \quad \text{etc.}$$

One proves, just as for M, that the R_α-modules M_α, determined by (7*a*) and (8*a*), are S_α-torsionfree $(\forall \alpha \in \mathrm{A})$. Summarizing we proved that

2.1. *The general subproduct* M, *defined by* (7) *and* (8), *has the following properties*:

(*i*) M *is an* S-*torsionfree* R-*module*;

(*ii*) (7) *and* (8) *determine* (*for each* $\alpha \in \mathrm{A}$), *systems* (7*a*) *and* (8*a*), *i.e. they determine* S_α-*torsionfree* R_α-*modules* M_α (*for each* $\alpha \in \mathrm{A}$).

We prove that M can be represented as a subdirect product $\mathrm{M} = \underset{\alpha}{\tilde{\times}}\, \mathrm{M}$, and—denoting the canonical projections $\mathrm{M} \to \mathrm{M}_\alpha$ by Π_α—that $\mathrm{Ker}\,(\Pi_\alpha) = \mathrm{Ann}_\mathrm{M}\,(\mathrm{R} \cap \mathrm{R}_\alpha)$, $\alpha \in \mathrm{A}$.

Proof. If $m = (m^{(i)})$ satisfies (7) and (8), then $m_\alpha = (m_\alpha^{(i)})$ is a solution of (7*a*) and (8*a*). If we map therefore $m \mapsto (\cdots, m_\alpha, \cdots)$, then it is clear that M is a subdirect product of the M_α $(\alpha \in \mathrm{A})$. We prove even that the decomposition $\mathrm{M} = \underset{\alpha \in \mathrm{A}}{\tilde{\times}}\, \mathrm{M}$ is the canonical decomposition corresponding with the canonical representation $R = \underset{\widetilde{\alpha}}{\times} R_\alpha$ of R. Therefore we prove that the kernels of the canonical projections $\Pi_\alpha : M \to M_\alpha$ are

$$\operatorname{Ker}(\Pi_\alpha) = \operatorname{Ann}_M (R \cap R_\alpha) = \operatorname{Ann}_M S_\alpha .$$

$$\operatorname{Ann}_M S_\alpha = \{m \in M \mid m(0, 0, \cdots, 0, r_\alpha, 0, \cdots) = 0 \quad \forall r_\alpha \in R \cap R_\alpha\} =$$
$$= \{(m^{(1)}, m^{(2)}, \cdots, m^{(i)}, \cdots) \in M \mid (\cdots, m^{(i)}, \cdots)(0, 0, \cdots, r_\alpha, 0, \cdots) = 0, \quad \forall r_\alpha \in R \cap R_\alpha\}$$

and i.e.

$$m_\alpha^{(i)} \in N_\alpha^{(i)} = \operatorname{Ann}_{M^{(i)}} S_\alpha \qquad (i \in I) .$$

Therefore $\operatorname{Ker}(\Pi_\alpha) = \operatorname{Ann}_M S_\alpha = \{m = (m^{(i)}) \mid m^{(i)} \in N_\alpha^{(i)};\ i \in I\}$, where

$$N_\alpha^{(i)} = \{m^{(i)} \in M^{(i)} \mid m^{(i)} \in \operatorname{Ann}_{M^{(i)}} S_\alpha\} .$$

LITERATURE

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[3] F. LOONSTRA (1977) - *Subproducts and subdirect products*, « Publ. Math. Debrecen », *24*, 129-137.

[4] F. LOONSTRA (1981) - *Special cases of subproducts*, « Rend. Sem. Mat. Univ. Padova », *65*, 175-185.