Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Robert W. Carroll

On the canonical development of Parseval formulas for singular differential operators

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **72** (1982), n.2, p. 65–70.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1982_8_72_2_65_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1982.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 13 febbraio 1982

Presiede il Presidente della Classe GIUSEPPE MONTALENTI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *On the canonical development of Parseval formulas for singular differential operators.* Nota di ROBERT CARROLL (*), presentata (**) dal Socio C. MIRANDA.

RIASSUNTO. — Per funzioni opportune f, g si ottiene una formula di Parseval $\langle R^Q, \mathcal{Q}f\, \mathcal{Q}g\rangle_\lambda = \langle \Delta_Q^{-\frac{1}{2}} f, \Delta_Q^{-\frac{1}{2}} g\rangle$ per operatori differenziali singolari di tipo dell'operatore radiale di Laplace–Beltrami. R^Q è una funzione spettrale generalizzata di tipo Marčenko e può essere rappresentata per mezzo di un certo nucleo della trasmutazione.

INTRODUCTION

The use of transmutation methods in studying Parseval formulas and eigenfunctions expansions for differential operators goes back to Marčenko, Naimark, *et al.* in the early 1950's. Subsequently Marčenko introduced the idea of a generalized spectral function to handle nonselfadjoint problems and provided an elegant framework involving transmutation and Paley-Wiener information to deal with expansion theorems and Parseval formulas for operators of the form $D^2 - q(x)$ in a unified manner (cf. [23]). Some aspects of this approach were extended by Gasymov (see e.g. [19]) to singular operators $Q_m^0(D) - q(x) = D^2 + ((2m+1)/x) D - q(x)$ for $l = m - \frac{1}{2}$ an integer. In [2; 3] we indicated how the basic Marčenko procedure could be extended

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in a context of general transmutation theory (cf. [1; 4; 5; 6; 7; 8; 9; 10; 11; 12; 13; 14]) to $Q_m^0(D) - q(x)$ for general $m \geq -\frac{1}{2}$ and this led in particular to an alternate derivation of Gasymov's results. The formulation in [2; 3] was phrased in a "canonical" manner in the expectation, predicted there, that it would extend to singular operators $\hat{Q} = Q^0 + \rho_Q^2 - q(x)$ where $Q^0 u = (\Delta_Q u')'/\Delta_Q$ is modeled on a radial Laplace-Beltrami operator in a rank one noncompact Riemannian symmetric space (cf. [5; 6; 8; 13; 14; 17; 18; 20; 25]—here $\rho_Q = \frac{1}{2} \lim \Delta_Q'/\Delta_Q$ as $x \to \infty$). This does in fact transpire and we sketch here some of these "canonical" results, the full and considerable details for which will appear in [4].

2. Basic framework. One can phrase suitable hypotheses on Δ_Q in various ways and we restrict ourselves here to singular problems (cf. [15; 16] for other situations). For example in [25] one takes $\Delta_Q(x) = x^{2m+1} C_Q(x)$, $m > -\frac{1}{2}$, where C_Q is an even C^∞ strictly positive function. Generally we also think of $\Delta_Q \uparrow \infty$ as $x \to \infty$ with $\Delta_Q'/\Delta_Q \downarrow 2\rho \geq 0$ as in [17]. Typical model situations are indicated in [18; 20] in the form $\Delta_Q = (e^x - e^{-x})^{2\alpha+1} \cdot (e^x + e^{-x})^{2\beta+1}$. On the other hand for $q(x)$ one assumes $q \in C^\infty$ is even and real in [25] while singularities of q are permitted in [17; 24]. For simplicity we will exclude strong singularities ($q \sim \beta^2/x^2$ near $x = 0$) in order to deal here with transforms based on spherical functions $\varphi_\lambda^Q(x)$ satisfying $\hat{Q}\varphi = -\lambda^2 \varphi$, $\varphi_\lambda^Q(0) = 1$, and $D_x \varphi_\lambda^Q(0) = 0$. In any case our hypotheses on $q(x)$ will be implicit, in requiring certain properties of a transmutation kernel $L(x, y)$ below; generally q is complex valued and if a certain finite number (l—unspecified here) of derivatives exist with suitable growth as $x \to 0$ then $L(x, y)$ wil be admissable (cf. [19; 24]).

Let us write now $\hat{P} = Q^0 + \rho_Q^2$ and $\hat{Q} = \hat{P} - q(x)$. Associated with $\hat{P}$ and $\hat{Q}$ we have spherical functions φ_λ^P and φ_λ^Q as above and one defines Jost solutions $\Phi_{\pm\lambda}^P$ for example as solutions of $\hat{P}\varphi = -\lambda^2 \varphi$ asymptotic to $\Delta_P^{-\frac{1}{2}}(x) \exp(\pm i\lambda x)$ as $x \to \infty$. It follows that $\varphi_\lambda^P(x) = c_P(\lambda) \Phi_\lambda^P(x) + c_P(-\lambda) \Phi_{-\lambda}^P(x)$ with $2 i\lambda c_P(\lambda) = \Delta_P(x) W(\varphi_\lambda^P, \Phi_{-\lambda}^P)$ ($W(f, g) = f' g - fg'$) and the spectral theory for $\hat{P}$ will be based on a measure $d\nu_P(\lambda) = (1/2\pi |c_P(\lambda)|^2) d\lambda = \hat{\nu}_P(\lambda) d\lambda$. Thus for suitable f set $\Omega_\lambda^P(x) = \Delta_P(x) \varphi_\lambda^P(x)$ with $\mathfrak{P}f(\lambda) = \langle f(x), \Omega_\lambda^P(x) \rangle = \int_0^\infty f(x) \Omega_\lambda^P(x) dx$ and $\mathfrak{P}^{-1} = \overline{\mathfrak{P}}$ where $\overline{\mathfrak{P}}F(x) = \int_0^\infty F(\lambda) \varphi_\lambda^P(x) d\nu_P(\lambda)$. We will assume here that $R_0(\lambda) = \hat{\nu}_P(\lambda)$ is known although there are techniques for "discovering" R_0 through a transmutation $B_P : D^2 \to \hat{P}$ (cf. [2; 4]). Similarly for $\hat{Q}$ we have a transform $\mathfrak{Q}f(\lambda) = \langle f(x), \Omega_\lambda^Q(x) \rangle$ as above and we set also $\mathcal{Q}g(\lambda) = \langle g(x), \varphi_\lambda^Q(x) \rangle$ (note that $\Delta_Q = \Delta_P$). The inversion theory for $\mathfrak{Q}$ is achieved through a generalized spectral function R^Q, which will be a distribution acting on a certain space of entire functions, such that for suitable $f, g\ (f = \Delta_Q \check{f}, g = \Delta_Q \check{g})$, a Parseval formula

$$(2.1)\qquad \langle R^Q, \mathcal{Q}f\, \mathcal{Q}g\rangle_\lambda = \langle R^Q, \mathfrak{Q}\check{f}\ \mathfrak{Q}\check{g}\rangle_\lambda = \langle \Delta_Q^{-\frac{1}{2}} f, \Delta_Q^{-\frac{1}{2}} g\rangle = \langle \Delta_Q^{\frac{1}{2}} \check{f}, \Delta_Q^{\frac{1}{2}} \check{g}\rangle$$

holds. Note formally if $\check{g} = \delta(x - y)/\Delta_Q(x)$ then $\mathfrak{Q}\check{g} = \varphi_\lambda^Q(y)$ and $\langle \Delta_Q^{\frac{1}{2}} \check{f}, \Delta_Q^{\frac{1}{2}} \check{g}\rangle = \check{f}(y)$. This can be made rigorous and leads to the inversion $\check{f}(y) = \langle R^Q, \mathfrak{Q}\check{f}(\lambda)\, \varphi_\lambda^Q(y)\rangle_\lambda = \overline{\mathfrak{Q}}\{\mathfrak{Q}\check{f}\}(y)$.

We take $B : \hat{P} \to \hat{Q}\ (\hat{Q}B = B\hat{P})$ to be the transmutation characterized by $B\varphi_\lambda^P = \varphi_\lambda^Q$ and emphasize that B can be determined by solving partial differential equations for example without any use of spectral information (cf. [1; 4; 21; 22; 24]). We think of such transmutations working on C^∞ functions for example and let $\mathcal{B} = B^{-1}$. One can express the B and $\mathcal{B}$ action through kernel formulas $Bf(y) = \langle \beta(y, x), f(x)\rangle$ and $\mathcal{B}g(x) = \langle \gamma(x, y), g(y)\rangle$ where β and γ are triangular in the sense that $\beta(y, x) = 0$ for $x > y$ and $\gamma(x, y) = 0$ for $y > x$ while $\gamma(x, y) = \delta(x - y) + L(x, y)$ and $\beta(y, x) = \delta(x - y) + K(y, x)$. In this connection our hypotheses on q are expressed through requiring that $\Delta_P(x)\, \hat{l}(x, y) = \Delta_P(x)\, L(x, y)\, \Delta_Q^{-1}(y)$ be continuous for $0 \leq y \leq x$; one defines also $\hat{l}(x) = \hat{l}(x, 0)$.

Next one defines $E_P = \{f ; \Delta_P^{\frac{1}{2}} f \in L^2\}\ (= E_Q)$ and $E_P^c = \{f \in E_P ;$ supp f is compact$\}$. Spaces such as E_P and $E_P' = \{f ; \Delta_P^{-\frac{1}{2}} f \in L^2\}$ form a natural framework for studying transmutation (cf. [1; 2; 4; 6]). Set $\mathbf{E}_P^c = \{f ; \Delta_P^{-\frac{1}{2}} f \in L^2;$ supp f is compact$\}$ and thus $\mathbf{E}_P^c \subset (E_P)'$ for example. One thinks of $\mathfrak{P}$ or $\mathfrak{Q}$ acting in E_P^c and $\mathcal{P}$ or $\mathcal{Q}$ acting in $\mathbf{E}_P^c\ (\mathcal{P}f(\lambda) = \langle f(x), \varphi_\lambda^P(x)\rangle)$. Set $\hat{E}_P^c = \mathfrak{P}E_P^c = \mathcal{P}\mathbf{E}_P^c$ with a scalar product $\langle \hat{f}, \hat{g}\rangle = \int_0^\infty \hat{f}(\lambda)\, \hat{g}(\lambda)\, \hat{\nu}_P(\lambda)\, d\lambda$ $(\mathcal{P}(\Delta f) = \mathfrak{P}f = \hat{f})$ and one thinks of E_P^c for example as a countable union space in the sense of Gelfand-Šilov; thus $E_P^c = \cup E_P^c(\sigma)$ where σ refers to supp $f \subset [0, \sigma]$ and $\hat{E}_P^c(\sigma) = \mathfrak{P}E_P^c(\sigma)$ has a Hilbert structure. By Paley-Wiener type results (cf. [17; 18; 19; 20; 25]) one can characterize $\hat{E}_P^c$ as a space of even entire functions $\hat{f}$ of exponential type (determined by $|\hat{f}(\lambda)| \leq c \exp(\sigma\, |Im\lambda|)$) with $\hat{\nu}_P^{\frac{1}{2}} \hat{f} \in L_\lambda^2$. We take W to be the space of even entire functions F of exponential type (as above) such that $\int_0^\infty |F(\lambda)|\, \hat{\nu}_P(\lambda)\, d\lambda < \infty$. Note that $W \subset \hat{E}_P^c$ and given $\hat{f}, \hat{g} \in \hat{E}_P^c$ it follows that $\hat{f}\hat{g} \in W$. Set $K = \overline{\mathfrak{P}}W$ and $\mathbf{K} = \mathbf{P}W = \Delta_P K\ (\mathbf{P} \sim \mathcal{P}^{-1})$ with the transported topological structure. Finally we recall the idea of a generalized translation S_x^y associated with $\hat{Q}$ (cf. [1; 2; 4; 21]); one has a formula

$$(2.2)\qquad S_x^y \check{f}(x) = \langle R^Q, \mathfrak{Q}\check{f}(\lambda)\, \varphi_\lambda^Q(x)\, \varphi_\lambda^Q(y)\rangle_\lambda .$$

3. Parseval formulas. In order to establish (2.1) we need various ingredients some of which are stated as lemmas below. First one has, with the notation of Section 2 (cf. [1; 2; 3; 4]).

LEMMA 3.1. *$\mathscr{P}B^* f = \mathscr{Q}f$ and $\mathscr{Q}\mathscr{B}^* g = \mathscr{P}g$. In particular $B^* : \mathbf{E}_P^c \to \mathbf{E}_P^c$ so that for $f \in \mathbf{E}_P^c$, $\mathscr{Q}f = \mathscr{P}B^* f \in \mathscr{P}\mathbf{E}_P^c = \hat{\mathbf{E}}_P^c$ and $\mathscr{Q}f\,\mathscr{Q}g \in W$ for $f, g \in \mathbf{E}_P^c$.*

LEMMA 3.2. *For $\check{f}, \check{g} \in \mathbf{E}_P^c$ and $f = \Delta_Q \check{f}$, $g = \Delta_Q \check{g}$ one has*

$$\langle S_x^y \check{f}(x), g(x) \rangle = \int_0^\infty S_x^y \check{f}(x)\, \check{g}(x)\, \Delta_Q(x)\, dx = \tag{3.1}$$

$$= \int_0^\infty S_x^y \check{g}(x)\, \check{f}(x)\, \Delta_Q(x)\, dx = \langle f(x), S_x^y \check{g}(x) \rangle .$$

The proof of (2.1) goes as follows. Let $\delta^n(x)$ be an approximation to the delta function $\delta(x)$ in $\mathscr{E}'$ where $\delta^n \in C_0^\infty$, $\delta^n \geq 0$, $\delta^n = 0$ near 0 and for $x \geq 1/n$, and $\int_0^\infty \delta^n(x)\, dx = 1$. One sets " experimentally "

$$S_x^y \delta_Q^n(x) = \langle R_n^\vee(\lambda), \varphi_\lambda^Q(x)\, \varphi_\lambda^Q(y) \rangle_\nu \tag{3.2}$$

where $\delta_Q^n(x) = \delta^n(x)/\Delta_Q(x)$. Using Lemma 3.2 one can show

LEMMA 3.3. *Let $f, g \in \mathbf{E}_P^c$ with g_k continuous, $g_k = \Delta_Q \check{g}_k$, $g_k \to g$ in $\mathbf{E}_P^c$. Then*

$$\langle f(y), \langle S_x^y \delta_Q^n(x), g_k(x) \rangle \rangle \to \langle f(y), \check{g}_k(y) \rangle \to \langle \Delta_Q^{-\frac{1}{2}} f, \Delta_Q^{-\frac{1}{2}} g \rangle . \tag{3.3}$$

Formally this says that $S_x^y \delta_Q(x) = \delta(x-y)/\Delta_Q(y)$

Hence the left side of (3.2) will lead to one side of (2.1). On the other hand the right side of (3.2), operating on $f(y)\, g(x)$, $f, g \in \mathbf{E}_P^c$, leads to a term $\Upsilon_n = \langle R_n^\vee(\lambda), \mathscr{Q}f(\lambda)\, \mathscr{Q}g(\lambda) \rangle_\nu = \langle R_n^\vee(\lambda)\, \hat{\nu}_P(\lambda), \mathscr{Q}f\, \mathscr{Q}g \rangle_\lambda$. Thus we think of $R_n = R_n^\vee \hat{\nu}_P \in W'$ (since $\mathscr{Q}f\,\mathscr{Q}g \in W$) and one wants to determine a distribution $R^Q \in W'$ to which R_n converges weakly in W' (so that $\Upsilon_n \to \langle R^Q, \mathscr{Q}f\,\mathscr{Q}g \rangle$). First note from (3.2) with $y = 0$ one obtains

$$\delta_Q^n(x) = \langle R_n^\vee, \varphi_\lambda^Q(x) \rangle_\nu \,;\, \mathscr{B}\delta_n^Q(y) = \langle R_n^\vee, (\mathscr{B}\varphi_\lambda^Q)(y) \rangle_\nu = \tag{3.4}$$

$$= \langle R_n^\vee, \varphi_\lambda^P(y) \rangle_\nu = \overline{\mathfrak{P}} R_n^\vee .$$

Consequently for $k \in \mathbf{K}$, $\mathscr{P}h = H \in W$ ($H \sim \mathscr{Q}f\,\mathscr{Q}g$), it follows that

$$\langle \overline{\mathfrak{P}} R_n^\vee, h \rangle = \langle R_n^\vee, \mathscr{P}h \rangle_\nu = \langle R_n, \mathscr{P}h \rangle_\lambda \acute{=} \langle \delta_Q^n, h \rangle + \tag{3.5}$$

$$+ \left\langle h(x), \int_0^x L(x, y)\, \delta_Q^n(y)\, dy \right\rangle$$

where we have used the decomposition $\gamma(x, y) = \delta(x - y) + L(x, y)$. Now since $h = \mathbf{P}H$

$$(3.6) \qquad \langle \delta_Q^n(x), h(x) \rangle = \langle \delta^n(x), h(x)/\Delta_P(x) \rangle = $$

$$= \int_0^\infty \mathscr{P}h(\lambda) \langle \delta^n(x), \varphi_\lambda^P(x) \rangle \hat{\nu}_P(\lambda)\, d\lambda =$$

$$= \langle \mathscr{P}h(\lambda), \hat{\nu}_P(\lambda) \langle \delta^n(x), \varphi_\lambda^P(x) \rangle \rangle_\lambda = \langle \mathscr{P}h(\lambda), R_0^n \rangle \to \langle \mathscr{P}h(\lambda), R_0 \rangle_\lambda$$

since $\langle \delta^n(x), \varphi_\lambda^P(x) \rangle \to 1$ suitably (recall $R_0 = \hat{\nu}_P$). For the second term in (3.5) we first write

$$\psi_n(x) = \Delta_P(x) \int_0^x L(x, y)\, \delta_Q^n(y)\, dy = \int_0^x \Delta_P(x)\, \hat{l}(x, y)\, \delta^n(y)\, dy .$$

One has $\psi_n(x) \to \Delta_P(x)\, \hat{l}(x)$ and $\left\langle h(x), \int_0^x L(x, y)\, \delta_Q^n(y)\, dy \right\rangle = \langle h\Delta_P^{-1}, \psi_n \rangle \to$ $\to \langle h(x), \hat{l}(x) \rangle$. Since $H = \mathscr{P}h \to h(x)/\Delta_P(x)$ is suitably continuous $\langle h, \hat{l} \rangle$ determines a distribution $R_q \in W'$ by the rule $\langle R_q, H \rangle_\lambda = \langle h, \hat{l} \rangle$. Similarly $\langle h\Delta_P^{-1}, \psi_n \rangle = \langle R_q^n, H \rangle$ and $R_q^n \to R_q$ weakly in W'. An explicit formula for R_q^n and R_q as distributions can be obtained by writing formally

$$(3.7) \qquad \langle R_q^n, H \rangle = \int_0^\infty h(x)\, \Delta_P^{-1}(x) \int_0^x \Delta_P(x)\, \hat{l}(x, y)\, \delta^n(y)\, dy\, dx =$$

$$= \int_0^\infty \int_0^x \hat{l}(x, y)\, \delta^n(y) \int_0^\infty \Omega_\lambda^P(x)\, H(\lambda)\, \hat{\nu}_P(\lambda)\, d\lambda\, dy\, dx =$$

$$= \left\langle H, \hat{\nu}_P \int_0^\infty \Omega_\lambda^P(x) \left\{ \int_0^x \hat{l}(x, y)\, \delta^n(y)\, dy \right\} dx \right\rangle .$$

Consequently

$$R_q^n = \hat{\nu}_P \int_0^\infty \Omega_\lambda^P(x) \langle \hat{l}(x, y), \delta^n(y) \rangle\, dx \qquad \text{and} \qquad R_q = \hat{\nu}_P \int_0^\infty \Omega_\lambda^P(x)\, \hat{l}(x)\, dx .$$

We note also from (3.4) that $R_n^\vee = \mathfrak{P}\mathscr{B}\delta_Q^n$ so formally $R^\vee = \mathfrak{P}\mathscr{B}\delta_Q$ yields $R = R_0 + R_q = \hat{\nu}_P R^\vee = \hat{\nu}_P \mathfrak{P} \{ \delta_Q + \langle L(x, y), \delta_Q(y) \rangle \} = \hat{\nu}_P \{ 1 + \mathfrak{P}\hat{l} \}$ in agreement with our other calculations. Thus

THEOREM 3.4. *Under the hypotheses indicated on* $L(x, y)$ *one has a Parseval formula* (2.1), $\langle R^Q, \mathscr{Q}f\, \mathscr{Q}g \rangle_\lambda = (\Delta_Q^{-\frac{1}{2}} f, \Delta_Q^{-\frac{1}{2}} g)$ *for* $f, g \in \mathbf{E}_P^c$, *where* $R^Q = R \in W'$ *can be written formally as* $R^Q = R_0 + \hat{\nu}_P \mathfrak{P}\hat{l}$.

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