Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Emilio Acerbi, Giuseppe Buttazzo, Nicola Fusco

Semicontinuity in L^∞ for polyconvex integrals

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **72** (1982), n.1, p. 25–28.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1982_8_72_1_25_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1982.

Calcolo delle variazioni. — *Semicontinuity in* L^∞ *for polyconvex integrals* (*). Nota di EMILIO ACERBI (**), GIUSEPPE BUTTAZZO (**) e NICOLA FUSCO (***), presentata (****) dal Socio C. MIRANDA.

RIASSUNTO. — Viene studiata la semicontinuità rispetto alla topologia di $L^\infty(\Omega\,;R^m)$ per alcuni funzionali del Calcolo delle Variazioni dipendenti da funzioni a valori vettoriali.

INTRODUCTION

The first results about the semicontinuity of functionals of the type

$$\int_\Omega f(x\,,u(x)\,,Du(x))\,dx\,, \tag{1}$$

where u is a vector-valued function, are due to Morrey [9] who proved that under certain regularity assumptions on the integrand f, the functional (1) is weakly* sequentially l.s. (lower semicontinuous) on $W^{1,\infty}(\Omega\,;R^m)$ if and only if for all $(x\,,s)$ the function $\xi \to f(x\,,s\,,\xi)$ is quasi-convex:

DEFINITION 1. *A continuous function* $\phi : R^{nm} \to R$ *is quasi-convex if for every open subset* Ω *of* R^n, *for every* $\xi \in R^{nm}$ *and every function* $w \in C_0^1(\Omega\,;R^m)$

$$\phi(\xi)\,\mathrm{meas}(\Omega) \le \int_\Omega \phi(\xi + Dw(x))\,dx\,.$$

The result of Morrey was generalized by Meyers [8] (in the case of integrals of any order) and by Acerbi and Fusco [1] who showed that if f is a Carathéodory function such that

$$0 \le f(x\,,s\,,\xi) \le a(x) + C(|s|^p + |\xi|^p) \qquad (p \ge 1)\,, \tag{2}$$

where a is non-negative and locally summable on R^n, and $C > 0$, then the functional (1) is weakly l.s. on $W^{1,p}(\Omega\,;R^m)$ if and only if the function $\xi \to f(x\,,s\,,\xi)$ is quasi-convex.

(*) This work has been supported by GNAFA (CNR).
(**) Scuola Normale Superiore - Piazza dei Cavalieri, 7 - 56100 Pisa.
(***) Istituto di Matematica - via Mezzocannone, 8 - 80100 Napoli.
(****) Nella seduta del 9 gennaio 1982.

Finally, we remark that several semicontinuity theorems have been also proved for convex functionals, even more general than the functional (1) (see e.g. [6]).

In the applications to nonlinear elasticity, one usually finds a particular class of quasi-convex functions, namely the class of polyconvex functions (see Ball [2], [3], Dacorogna [5]):

DEFINITION 2. *A function* $\phi : \mathrm{R}^{nm} \to \mathrm{R}$ *is polyconvex if there exists a convex function* ψ *such that for every* $m \times n$ *matrix* A

$$\phi(\mathrm{A}) = \psi(\mathrm{XA}),$$

where XA *denotes the vector whose components consist of all the subdeterminants of the matrix* A.

We remark here that if $\xi \to f(x, s, \xi)$ is for all (x, s) a non-negative polyconvex function, the semicontinuity theorem proved in [1] holds without the growth condition (2).

In general, if $1 \le p < \infty$, the integrals of polyconvex functions are not l.s. with respect to the topology of $\mathrm{L}^p(\Omega, \mathrm{R}^m)$ as one can already see for the functional $(m = n)$

$$\int_\Omega |\det \mathrm{D}u(x)| \, \mathrm{d}x .$$

For it is possible to construct a sequence (u_h) such that $\det \mathrm{D}u_h \equiv 0$, but (u_h) converges in L^p, for every $p < \infty$, to the function $u(x) = x$. Nevertheless this functional turns out to be l.s. with respect to the topology of $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega ; \mathrm{R}^n)$.

This example shows why for the integrals of polyconvex functions it is not interesting to study the lower semicontinuity with respect to the topology of $\mathrm{L}^p(\Omega ; \mathrm{R}^m)$ (with $p < \infty$).

On this subject the situation in the scalar case is completely different. Indeed when u is a real-valued function and f satisfies (2), the semicontinuity of the functional (1) with respect to the topology of $\mathrm{L}^\infty(\Omega)$ is equivalent to the semicontinuity in the topology of $\mathrm{L}^p(\Omega)$, at least on $\mathrm{W}^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ (see Carbone-Sbordone [4]).

In this paper we state some results on the semicontinuity in the topology of $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega ; \mathrm{R}^m)$ of certain integrals of polyconvex functions.

By the previous remark it is clear that the results stated here generalize the semicontinuity theorems proved in the scalar case in [10], [11]. At any rate they apply to the important class of parametric integrals (widely studied in [7], [10]) and to the most representative examples of polyconvex integrals occurring in nonlinear elasticity.

RESULTS

Let Ω be an open subset of R^n; if u is a function from Ω to R^m, let Du denote the matrix of its derivatives, and let $X^0 u$ denote the vector whose $\binom{m}{n}$ components are the subdeterminants of Du of order n. In the following theorem we specialize to the case $m = n + 1$.

THEOREM 1. *Let* $f : \Omega \times R^{n+1} \times R^{n+1} \to [0, +\infty]$ *satisfy*:

(3) *for every* $(x, s) \in \Omega \times R^{n+1}$ *the function* $\xi \to f(x, s, \xi)$ *is convex and lower semicontinuous, and* $f(x, s, 0) = 0$;

(4) *for every* $\Sigma \subset\subset \Omega \times R^{n+1}$ *there exists a continuous function* $\omega_\Sigma : R_+ \times R_+ \to R_+$, *vanishing in* $(0, 0)$, *such that for every* $(x, s), (y, t) \in \Sigma$ *and every* $\xi \in R^{n+1}$

$$| f(x, s, \xi) - f(y, t, \xi)| \leq \omega_\Sigma (| x - y |, | s - t |) [1 + f(x, s, \xi)] .$$

Then the functional $\int_\Omega f(x, u(x), X^0 u(x))\, dx$ *is l.s. on the space* $W^{1,n}_{loc}(\Omega ; R^{n+1}) \cap C(\Omega ; R^{n+1})$, *endowed with the topology of* $L^\infty_{loc}(\Omega ; R^{n+1})$.

An interesting example of integrand satisfying (3), (4) is

$$f(x, s, \xi) = a(x, s)\, \phi(\xi)$$

where ϕ is convex and lower semicontinuous, $\phi(0) = 0$, and a is a continuous function with some positive lower bound.

We remark that the foregoing result is still valid for the functional $\int_\Omega f(x, u(x), \det Du(x))\, dx$, where u is a function from Ω to R^n and $f : \Omega \times R^n \times R \to [0, +\infty]$ satisfies conditions analogous to (3), (4).

More generally, one may consider functionals of the type

$$\int_\Omega f(x, u(x), X^+ u(x))\, dx ,$$

where $X^+ u$ denotes the vector whose components are the absolute values of all the subdeterminants of Du; in addition let $r(n, m)$ denote the dimension of the vector $X^+ u$.

THEOREM 2. *Let* $f : \Omega \times R^m \times R_+^{r(n,m)} \to [0, +\infty]$ *satisfy*:

(5) *for every* $(x, s) \in \Omega \times R^m$ *the function* $\xi \to f(x, s, \xi_+)$ *is convex and lower semicontinuous (here* ξ_+ *denotes the vector of the absolute values of the components of* ξ);

(6) *for every* $\Sigma \subset\subset \Omega \times R^m$ *there exists a continuous function* $\omega_\Sigma : R_+ \times R_+ \to R_+$, *vanishing in* $(0, 0)$, *such that for every* $(x, s), (y, t) \in \Sigma$ *and for every* $\xi_+ \in R_+^{r(n,m)}$

$$|f(x, s, \xi_+) - f(y, t, \xi_+)| \leq \omega_\Sigma(|x-y|, |s-t|)[1 + f(x, s, \xi_+)].$$

Then the functional $\int_\Omega f(x, u(x), X^+ u(x))\, dx$ *is l.s. on the space* $W^{1,n}_{loc}(\Omega; R^m) \cap C(\Omega; R^m)$, *endowed with the topology of* $L^\infty_{loc}(\Omega; R^m)$.

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