Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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The linear-quadratic optimal control problem for delay differential equations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **71** (1981), n.6, p. 156–161.

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<http://www.bdim.eu/item?id=RLINA_1981_8_71_6_156_0>

Analisi matematica. — *The linear-quadratic optimal control problem for delay differential equations.* Nota di GABRIELLA DI BLASIO (*) (**), presentata (***) dal Corrisp. E. VESENTINI.

RIASSUNTO. — In questo lavoro si considera il problema del controllo ottimo per un'equazione lineare con ritardo in uno spazio di Hilbert, con costo quadratico. Si dimostra che il problema della sintesi si traduce in una equazione di Riccati in uno opportuno spazio prodotto e si prova che tale equazione ammette un'unica soluzione.

1. INTRODUCTION

Let X be a Hilbert space and let $A : D_A \subset X \to X$ be a linear operator. We shall be concerned with the linear-quadratic optimal control problem associated with the delay differential equation

$$x'(t) = Ax(t) + Ax(t-r), \qquad t \geq 0 \tag{1}$$

where $r > 0$ is given. Problems of this kind have been extensively studied by Delfour and Mitter when A is a bounded operator on X. In this case it is possible to formulate the control problem associated with (1) in the product space $X \times L^2(-r, 0; X)$ as a problem without delay (see e.g. [5] and the references therein). This approach seems to be no longer useful if A is unbounded.

In this paper we consider the case where A is the infinitesimal generator of an analytic semigroup and show that the product-space approach can be used by choosing some suitable interpolation space between D_A and X.

2. THE SEMIGROUP GENERATED BY THE DELAY EQUATION

We begin with some notation and terminology. Let $a < b$ be real numbers and let E be a Hilbert space. By $L^2(a, b; E)$ we shall denote the Hilbert space of square integrable functions from $]a, b[$ into E and by $W^1(a, b; E)$ the set of all functions of $L^2(a, b; E)$ whose distributional derivative belongs to $L^2(a, b; E)$ In what follows we shall consider a Hilbert space X and a linear operator $A : D_A \subset X \to X$ which generates an analytic semigroup on X. By $D_A(\theta, p)$

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(**) Work done under the auspices of G.N.A.F.A.
(***) Nella seduta del 12 dicembre 1981.

we denote the real interpolation space $T_0^1(p, 1-\theta-p^{-1}; D_A; p, 1-\theta-p^{-1}; X)$ between D_A and X obtained by the traces' method (see Lions-Peetre [8] cap. VI). Further we denote by F the Hilbert space $F = D_A(1/2, 2)$ by Y the Hilbert space $Y = F \times L^2(-r, 0; D_A)$ with the inner product

$$\langle x, y \rangle_Y = \langle x_0, y_0 \rangle_F + \int_{-r}^{0} \langle x_1(s), y_1(s) \rangle_{D_A} ds$$

for each $x = (x_0, x_1)$ and $y = (y_0, y_1)$ in Y.

The following theorem holds

THEOREM 1. *For each* $h = (h_0, h_1) \in Y$ *and* $T > 0$ *there exists a unique* $x \in W^1(0, T; X) \cap L^2(0, T; D_A)$ *solution of the following problem*

$$\begin{cases} x'(t) = Ax(t) + Ax(t-r), & \text{a.e. on} \quad]0, T[\\ x(t) = h_1(t) \quad \text{a.e. on} \quad]-r, 0[, & x(0) = h_0. \end{cases} \tag{2}$$

Moreover there exists $c(T)$ such that

$$\| x \|_{L^2(0,T;D_A)} \quad , \quad \| x' \|_{L^2(0,T;X)} \leq c(T) \| h \|_Y . \tag{3}$$

Proof. Let $T \leq r$; then (2) reduces to the solution of the following problem

$$\begin{cases} x'(t) = Ax(t) + Ah_1(t-r) \\ x(0) = h_0 \end{cases}$$

and the theorem can be proved by using the results of Da Prato–Grisvard [4] (see also Lions–Magenes [7]). If $T > r$ the result follows by iteration.

Now let us denote by $x(t; h)$ the value at time $t \geq 0$ of the solution of (2) and set $x_t(h)(\theta) = x(t + \theta, h)$ for a.e. θ in $]-r, 0[$. Furthermore let us denote by $S(t)$, $t \geq 0$, the family of operators on Y defined as

$$S(0) h = h$$

$$S(t) h = (x_t(h)(0), x_t(h)), \qquad \text{for} \quad t > 0.$$

We have

THEOREM 2. $S(t)$ *is a* C_0-*semigroup on* Y, *i.e.*

(*i*) $S(t) \in \mathscr{L}(Y)$ (1), *for each* $t \geq 0$

(*ii*) $S(0) = I$ (identity operator)

(*iii*) $S(t+s) h = S(t) S(s) h$, for each $s, t \geq 0$

(*iv*) $\lim_{t \to 0} S(t) h = h$, for each $h \in Y$.

(1) If E_1 and E_2 are Banach spaces we denote by $\mathscr{L}(E_1, E_2)$ the Banach space of linear bounded operators from E_1 into E_2 with the usual norm. Further we set $\mathscr{L}(E_1) = \mathscr{L}(E_1, E_1)$.

Proof. Assertion (*i*) can be proved by using (3). Assertion (*ii*) is evident and assertion (*iii*) is a consequence of the uniqueness of the solution of (2). Concerning (*iv*) we have

$$\text{(4)} \qquad \| S(t)\, h - h \|_Y^2 = \| x_t(h)(0) - h_0 \|_F^2 + \| x_t(h) - h_1 \|_{L^2(-r,0;D_A)}^2 .$$

Now the first term of the right-hand side of (4) tends to zero, as $t \to 0$, since x is continuous from $[0, T]$ into F (see [7] cap. 3). Moreover the last term tends to zero since the solution of (2) belongs to $L^2(-r, T; D_A)$.

Now let $\Lambda : D_\Lambda \subset Y \to Y$ be the operator defined as

$$D_\Lambda = \{(h_0, h_1) \in Y : h_1' \in L^2(-r, 0; D_A),\ h_0 = h_1(0),\ Ah_0 + Ah_1(-r) \in F\}$$

$$\Lambda h = (Ah_0 + Ah_1(-r), h_1').$$

The following theorem holds

THEOREM 3. *The operator* Λ *is the infinitesimal generator of the semigroup* $S(t)$.

Proof. Let $h \in D_\Lambda$ and let y be the solution of the problem

$$\begin{cases} y'(t) = Ay(t) + Ay(t-r), & \text{a.e. on }]0, T[\\ y(t) = h_1'(t) \quad \text{a.e. on }]-r, 0[, & y(0) = Ah_0 + Ah_1(-r) \end{cases}$$

It can be proved that $y = x'$, where x is the solution of (2). Therefore we have $x \in W^1(-r, T; D_A)$ and $x' \in W^1(0, T; X)$. Moreover

$$\left\| \frac{S(t)\, h - h}{t} - h \right\|_Y^2 = \left\| \frac{x(t; h) - x(0; h)}{t} - x'(0) \right\|_F^2 + \left\| \frac{x_t(h) - h_1}{t} - h_1' \right\|_{L^2(-r,0;D_A)}^2 .$$

Now the first term of the right-hand side tends to zero, as $t \to 0$, since x' is continuous from $[0, T]$ into F. Moreover the last term tends to zero since x belongs to $W^1(-r, T; D_A)$. Hence we have proved that if $h \in D_\Lambda$ then

$$\lim_{t \to 0} \frac{S(t)\, h - h}{t} = \Lambda h .$$

Therefore if we denote by Λ' the infinitesimal generator of $S(t)$ we have $\Lambda' \supset \Lambda$. Finally we get $\Lambda' = \Lambda$ as it can be checked that for λ sufficiently large we have $(\lambda I - \Lambda) D_\Lambda = Y$.

3. THE LINEAR–QUADRATIC OPTIMUM CONTROL PROBLEM

Let U be a Hilbert space and let G , K , N and B be linear operators satisfying the following properties

(*i*) $G, K \in \mathscr{L}(F)$ are symmetric and positive

(*ii*) $N \in \mathscr{L}(U)$ is symmetric and strictly positive

(*iii*) $B \in \mathscr{L}(U, F)$.

We shall consider the problem of minimizing the functional

$$\tfrac{1}{2}\langle Gx(T), x(T)\rangle_F + \tfrac{1}{2}\int_0^T (\langle Kx(t), x(t)\rangle_F + \langle Nu(t), u(t)\rangle_U)\, dt$$

over all $x \in W^1(0, T; X) \cap L^2(0, T; D_A)$ and $u \in L^2(0, T; U)$ satisfying the system

$$x'(t) = Ax(t) + Ax(t-r) + Bu(t), \quad \text{a.e. on }]0, T[$$
$$x(t) = h_1(t) \quad \text{a.e. on }]-r, 0[, \quad x(0) = h_0 .$$

To study this problem we introduce the operators $\overline{G}, \overline{K} : Y \to Y$ and $\overline{B} : U \to Y$ defined as

$$\overline{G}z = (Gz_0, 0) \quad , \quad \text{for each } z = (z_0, z_1) \in Y$$
$$\overline{K}z = (Kz_0, 0) \quad , \quad \text{for each } z = (z_0, z_1) \in Y$$
$$\overline{B}u = (Bu, 0) \quad , \quad \text{for each } u \in U .$$

From the results of the preceding section we see that the control problem can be reformulated as follows (see [5]): minimize

$$\tfrac{1}{2}\langle \overline{G}z(T), z(T)\rangle_Y + \tfrac{1}{2}\int_0^T (\langle \overline{K}z(t), z(t)\rangle_Y + \langle Nu(t), u(t)\rangle_U)\, dt \tag{5}$$

over all $z \in L^2(0, T; Y)$ and $u \in L^2(0, T; U)$ satisfying (in the mild sense)

$$\begin{aligned} z' &= \Lambda z + \overline{B}u \\ z(0) &= h . \end{aligned} \tag{6}$$

We may now apply the results concerning the classic linear-quadratic optimum control problem. Thus we have that there exists a unique solution

$\bar{z}$, $\bar{u}$ of (6) that minimize (5). Moreover $\bar{u}$ admits the feedback rapresentation

$$\bar{u}(t) = -N^{-1}\bar{B}^* P(T-t)\bar{z}(t)$$

where the asterisk denotes the adjoint and $P(t) \in \mathscr{L}(Y)$ is the solution of the Riccati equation

$$(7) \qquad \begin{cases} P' = \Lambda^* P + P\Lambda - PBN^{-1}\bar{B}^* P + \bar{K} \\ P(0) = \bar{G}. \end{cases}$$

We shall study the following integrated form of (7)

$$(8) \qquad P(t) z = S^*(t)\bar{G} S(t) z + \int_0^t S^*(t-s)(-P(s)\bar{B} N^{-1}\bar{B}^* P(s) + \bar{K}) S(t-s) z \, ds$$

where $S^*(t)$ is the semigroup generated by Λ^*. The following theorem holds

THEOREM 4. *For each* $T > 0$ *there exists a unique strongly continuous symmetric and positive* $P(t)$ *solution of equation* (8). *Moreover we have*

$$(9) \qquad \| P(t) \|_{\mathscr{L}(Y)} \leq M^2 (\| \bar{G} \|_{\mathscr{L}(Y)} \exp(2\omega t) + \| \bar{K} \|_{\mathscr{L}(Y)} \int_0^t \exp(2\omega(t-s)) \, ds)$$

where M and ω are such that $\| S(t) \|_{\mathscr{L}(Y)} \leq M \exp(\omega t)$.

Proof. The existence of local solutions of (8) can be proved by using the results of Da Prato [2]. Assertion (9) and global existence follows from noting that

$$\langle P(t) z, z \rangle_Y \leq \langle \bar{G} S(t) z, S(t) z \rangle_Y + \int_0^t \langle \bar{K} S(t-s) z, S(t-s) z \rangle_Y \, ds$$

which implies (9) since P is positive. Finally the uniqueness of the solution of (8) can be proved by using standard arguments.

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