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The point on the simple Molodensky's problem

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Geodesia. — *The point on the simple Molodensky's problem.* Nota di FERNANDO SANSÒ (**), presentata (*) dal Socio L. SOLAINI.

RIASSUNTO. — Il problema di Molodensky, in approssimazione sferica è detto « semplice » perchè può essere trasformato da problema di derivata obliqua a problema di Dirichlet per l'operatore di Laplace. Tale problema è accuratamente analizzato in questa Nota, con particolare riguardo alla generalizzazione delle condizioni di regolarità soddisfatte dal contorno S, sufficienti a garantire l'esistenza di una soluzione fisicamente accettabile.

INTRODUCTION

A deep critical comment on the author's paper [3] has raised an interesting question: how is it possible to define, may be in a weak sense, an oblique derivative problem (as Molodensky's problem is) for the Laplace operator, with a boundary S so irregular as to admit conical points (in analytical terminology we say a boundary satisfying a cone condition)? This is not a weird question, since to prove existence and uniqueness of the solution of Molodensky's problem for boundaries of this kind is a target at which we must aim, in order to get a theoretical frame in reasonable agreement with physical data. The solution seems to be rather difficult for a general oblique derivative problem, however it becomes much easier for the simple Molodensky's problem, due to its essential equivalence with a Dirichlet's problem. For this particular case we fix hereafter the point of the author's theoretical investigations.

POSITION OF THE PROBLEM

We aim at proving the existence and uniqueness of the solution of the simple Molodensky's problem formulated as: find T and $(a_j\,,\ j=-1\,,0\,,1)$ such that

$$
(1)\qquad \begin{cases} \Delta \mathrm{T} = 0 \quad \text{in} \quad \Omega \\ \mathrm{T} + \dfrac{1}{2}\, r\, \dfrac{\partial \mathrm{T}}{\partial r} = f + \sum_{-1}^{n}{}_j\, a_j\, \mathrm{A}_j \quad \text{on} \quad \mathrm{S} \\ \qquad\qquad \left(\mathrm{A}_j = \dfrac{\mathrm{Y}_{1j}(\varphi\,,\lambda)}{r^2} = \text{spherical harmonics of 1st order}\right) \\ \mathrm{T} = \dfrac{b}{r} + 0\,(r^{-3}) \qquad r \to \infty \end{cases}
$$

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under the following fairly general conditions

(2) $-\ \Omega$ *is a* $\mathcal{N}^{0,1}$ starshaped domain, i.e. $S \equiv \{r(\alpha)\, \boldsymbol{e}_r(\alpha) : \alpha \in \sigma\}$ $r(\alpha)$ is Lipschitz continuous, what entails the existence a.e. on S of the normal $\boldsymbol{n}(\alpha)$, and the essential boundedness of first derivatives of $r(\alpha)$: a domanin satisfying a cone condition is a finite union of $\mathcal{N}^{0,1}$ domains, as quoted in Necas [2], on page 71.

(3) $-\ f$ is an arbitrary function of $H^{1/2}(S)$.

We can prove that a unique solution of (1) exists, belonging to a suitably closed sub-space of $H^{1,2}(\Omega)$: the solution T will be such as to satisfy the boundary relation

$$T + \frac{1}{2} r \frac{\partial T}{\partial r} = f + \sum_{-1}^{1}{}_j\, a_j\, A_j$$

in the rather classical sense that the function $T + \frac{1}{2} r(\partial T/\partial r)$ considered as a spatial function belongs to $H^{1,2}(\Omega)$ and its trace (in $H^{1/2}(S)$) equals $f + \Sigma_j\, a_j\, A_j$.

Remark: this statement could also be seen in the sense that given a sequence S_n of surfaces in Ω, tending towards S in the $\mathcal{N}^{0,1}$ sense (i.e. $r_n(\alpha) \rightarrow r(\alpha)$ in $C^{0,1}(\sigma)$), $T + \frac{1}{2} r(\partial T/\partial r)|_{S_n} \rightarrow T + \frac{1}{2} r(\partial T/\partial r)|_S$ in $H^{1/2}(S)$).

Construction of the solution space

Let us start with the linear space $\mathcal{H}(\Omega)$ of all the harmonic functions in Ω regular at infinity. Now we consider a sub-space

$$H\,H^{1,2}(\Omega) = \left\{T\,,\ T \in \mathcal{H}(\Omega)\,,\ \int_{\Omega} |\nabla T|^2\, d\Omega < +\infty\right\}\ :$$

by dint of Harnak theorem this sub-space is a closed Hilbert space.

Let us now introduce a large sphere S_0, of radius R_0, enclosing S: we define in $H\,H^{1,2}(\Omega)$ the operator

$$P_1\, T = \sum_{-1}^{1}{}_j \left(\int_{S_0} Y_{1j}\, T\, dS_0\right) Y_{1j}/r^2\,. \tag{4}$$

It is straightforward to verify that

$$P_1^2\, T = P_1\, T$$

thus qualifying P_1 as a projection operator: P_1 is generally non orthogonal (unless S itself is a sphere), and its range is the sub-space of $H\,H^{1,2}(\Omega)$ spanned by (A_j). The complementary sub-space, i.e. the range of the projector $I - P_1$, which is also a closed sub-space fo $H\,H^{1,2}(\Omega)$ will be called $H'\,H^{1,2}(\Omega)$.

It is very easy to verify that the functions of $H' H^{1,2}(\Omega)$ are characterized by the asymptotic condition

$$T = (b/r) + 0\,(r^{-3}) \qquad (b = \text{suitable constant}).$$

Now let us introduce the operator

$$B = 1 + \tfrac{1}{2}\, r\,(\partial/\partial r) :$$

we define the following variety in $H' H^{1,2}(\Omega)$

$$V = \{T\,;\, T \in H' H^{1,2}(\Omega)\,,\, \| BT \|_{H^{1,2}(\Omega)} < +\infty\}\,. \tag{5}$$

Since $\Delta B = (1 + B)\,\Delta$ holds, we have that if $u = BT$, $T \in H' H^{1,2}(\Omega)$, then u is harmonic too in Ω, so that for $T \in V$, u belongs to $H\, H^{1,2}(\Omega)$.

Moreover from the fact that $P_1 T = 0$, i.e.

$$\int_{S_0} T\, Y_{1j}\, dS_0 = R_0^2 \int_\sigma T Y_{1j}\, d\sigma = 0\,,$$

and from

$$u = T + \tfrac{1}{2}\, r\,(\partial T/\partial r)\,,$$

we derive

$$\int_{S_0} u Y_{1j}\, dS_0 = R_0^2 \int_\sigma u Y_{1j}\, d\sigma = R_0^2 \left(\int_\sigma T Y_{1j}\, d\sigma + \frac{1}{2}\, R_0 \frac{\partial}{\partial R_0} \int_\sigma T Y_{1j}\, d\sigma \right) = 0 \Rightarrow P_1 u = 0\,.$$

We conclude then that $u = TB \in H' H^{1,2}(\Omega)$ for every $T \in V$: in other words V is the domain of the operator B, considered as an unbounded transformation of $H' H^{1,2}(\Omega)$ into itself.

We give to V the Hilbert space structure deriving from the norm choice

$$\| T \|_V = \| BT \|_{H' H^{1,2}(\Omega)}\,. \tag{6}$$

We must show that (7) is a true norm, i.e. essentially that

$$\| T \|_V = 0 \;\rightarrow\; T = 0\,.$$

But if

$$\| T \|_V = \| BT \|_{H' H^{1,2}(\Omega)} = 0$$

we have

$$T + \tfrac{1}{2}\, r\,(\partial T/\partial r) = 0 \qquad \text{a.e. in } \Omega$$

or

$$\partial/\partial r\,(r^2\, T) = 0 \qquad \text{a.e. in } \Omega$$

i.e.

$$T = g(\alpha)/r^2 :$$

recalling that $T \in H' H^{1,2}(\Omega)$ implies $T = b/r + O(r^{-3})$ we see that $g(\alpha) = 0$ and $T = 0$, q.e.d. In other words we have proved that B is an injective operator from V into $H' H^{1,2}(\Omega)$.

Now we want to show that B is onto $H' H^{1,2}(\Omega)$: this being true we may conclude that B transforms isometrically V onto $H' H^{1,2}(\Omega)$ and consequently V is closed under the norm assignement (6), since $H' H^{1,2}(\Omega)$ is complete.

To this aim let us take any $u \in H' H^{1,2}(\Omega)$ and search for a $T \in V$ such that

$$T + \tfrac{1}{2} r (\partial T/\partial r) = u . \tag{7}$$

We split the problem in two parts: first we solve (7) in the domain Ω_0, the exterior to the sphere S_0, showing that T belongs to $H' H^{1,2}(\Omega_0)$; subsequently after having verified that T is harmonic in Ω, we prove that $T \in H^{1,2}(\Omega \setminus \Omega_0)$ so that, summarizing we have $T \in H^{1,2}(\Omega)$, $\Delta T = 0$ in Ω, $T = b/r + O(r^{-3})$, i.e. $T \in H' H^{1,2}(\Omega)$. As for the first part, let us consider the series representation

$$u = \sum_{0}^{+\infty}{}_{n \neq 1} \sum_{-n}^{n}{}_m u_{nm} \left(\frac{R_0}{r}\right)^{n+1} Y_{nm}(\alpha)$$

which is certainly converging (even uniformly) in $\Omega_0 \cup S_0$ since $u \in H^{1,2}(\Omega)$, $\Omega \supset \bar{\Omega}_0$. To say that $u \in H^{1,2}(\Omega_0)$ amounts to verify that

$$\sum_{u \neq 1} \sum_{-n}^{n}{}_m n u_{nm}^2 < +\infty , \tag{8}$$

as it is easily realized by a direct computation of $\int_{\Omega_0} |\nabla T|^2 d\Omega_0$ in spherical coordinates.

Now let us define the function

$$T = -2 \sum_{n \neq 1} \sum_{-n}^{n}{}_m \frac{u_{nm}}{n-1} \left(\frac{R_0}{r}\right)^{n+1} Y_{nm}(\alpha) : \tag{9}$$

as it is apparent, because of (8), we have $\| T \|_{H^{1,2}(\Omega)}$, $\| \partial T/\partial r \|_{H^{1,2}(\Omega)} < +\infty$ too. Differentiating under the summation sign then, we verify that

$$BT = u .$$

Moreover T is manifestly harmonic in Ω_0 and the asymptotic relation $T = b/r + O(r^{-3})$ is fulfilled: whence $T \in H' H^{1,2}(\Omega_0)$ as we wanted to prove.

Now let us note that since T has been defined for $r \geq R_0$, a unique solution of (7) is defined also in $\Omega \setminus \Omega_0$ by direct integration through the formula

$$T(r, \alpha) = \frac{R_2^0}{r^2} T(R_0, \alpha) - \frac{2}{r^2} \int_r^{R_0} ds\, s u(s, \alpha) . \tag{10}$$

Relation (10) shows that $T \in L^1_{loc}(\Omega \setminus \Omega_0)$, so that ΔT exists at least in the distribution sense: subsequently we can say that

$$0 = \Delta u = \Delta BT = (1 + B)\,\Delta T \qquad \text{in } \Omega\,. \tag{11}$$

The equation

$$(1 + B) f = 2f + \tfrac{1}{2}\, r\,(\partial f/\partial r) = 0$$

has the integrating factor r^3, and is equivalent to

$$\partial/\partial r\ (r^4 f) = 0$$

which has the general solution

$$f = g(\alpha)/r^4 \qquad\qquad (g(\alpha) \text{ arbitrary}).$$

From (11) then we derive

$$\Delta T = g(\alpha)/r^4\,,$$

but since $\Delta T = 0$ for $r \geq R_0$ we have necessarily

$$\Delta\, T = 0 \qquad \text{in } \Omega. \tag{12}$$

Now let us come to the last point: from (10) it is clear that $T \in H^{1,2}(\Omega \setminus \Omega_0)$ if $\int_r^{R_0} ds\, su(s, \alpha)$ does, or equivalently if the functions

$$\partial_j \int_r^{R_0} dS\, su\left(s\frac{x_1}{r}, s\frac{x_2}{r}, s\frac{x_3}{r}\right) = -\, x_i\, u(r, \alpha) + \tag{13}$$

$$+ \int_r^{R_0} ds\, s\, \partial_k u(s, \alpha)\left[\delta_{ki} - \frac{x_k\, x_i}{r^2}\right]\frac{s}{r} = -x_j u + \frac{1}{r}\int_r^{R_0} ds\, s^2\,(\partial_i u) +$$

$$+ \frac{x_i}{r^2}\int_r^{R_0} ds\, s^2\left(\frac{\partial u}{\partial r}\right) = -\frac{x_i}{r^2}\, R_0^2\, u(R_0, \alpha) + \frac{1}{r}\int_r^{R_0} ds\, s^2\,(\partial_i u)(s, \alpha) +$$

$$+ \frac{2x_i}{r^2}\int_r^{R_0} ds\, su(s, \alpha)\,,$$

are of class L^2 in $\Omega \setminus \Omega_0$.

Since the first term in (13) is trivially in $L^2(\Omega \setminus \Omega_0)$, let us come to the second: we have

$$\int_\sigma d\alpha \int_{r(\alpha)}^{R_0} dr\, r^2 \left\{ \frac{1}{r} \int_r^{R_0} ds\, s^2 (\partial_i u)(s, \alpha) \right\}^2 \leq R_0 \int_\sigma d\alpha \left\{ \int_{r(\alpha)}^{R_0} ds\, s^2 (\partial_i u)(s, \alpha) \right\}^2 \leq$$

$$\leq \frac{R_0^4}{3} \int_\sigma d\alpha \int_{r(\sigma)}^{R_0} ds\, s^2 [\partial_i u(s, \alpha)]^2 \leqq \frac{R_0^4}{3} \| u \|^2_{H^{1,2}(\Omega \setminus \Omega_0)} < +\infty .$$

As for the last term, in an analogous way it is proved that

$$\int_\sigma d\alpha \int_{r(\alpha)}^{R_0} dr\, r^2 \left\{ \frac{2 x_i}{r^2} \int_r^{R_0} ds\, su(s, \alpha) \right\}^2 \leq 4 R_0^2 \| u \|^2_{L^2(\Omega \setminus \Omega_0)} \leqq \text{const} \| u \|^2_{H^{1,2}(\Omega \setminus \Omega_0)}$$

q.e.d.

Summarizing, we can say that the variety V endowed with the norm (6) is a Hilbert space.

Solution of the problem

Having suitably built up the space V, we can now formalize our problem (1) in a different manner: we can state that we are searching for $T \in V$ and three constants (a_j), such that

$$BT - \sum_{-1}^{1}{}_j\, a_j A_j \big|_S = f \in H^{1/2}(S) . \tag{14}$$

Since $BT - \sum_{-1}^{1}{}_j\, a_j A_j$ is a harmonic function, by the uniqueness of the solution of Dirichlet problem, (14) is equivalent to

$$BT - \sum_{-1}^{1}{}_j\, a_j A_j = u , \tag{15}$$

where u is the harmonic function agreeing with f on S. Since S is $\mathcal{N}^{0,1}$ and $f \in H^{1/2}(S)$ it is known (Necas [2]) that u exists and is unique in $H\,H^{1,2}(\Omega)$. In other words u is the unique element of $H\,H^{1,2}(\Omega)$ corresponding to the trace f on S: the correspondence between $H\,H^{1,2}(\Omega)$ and $H^{1/2}(S)$ is one to one and even an isometry for a suitable definition of the norm (cfr. Miranda [1] page 170).

Whence we have reduced our problem to: find $T \in V$, $(a_j) \in R^3$ such that for any $u \in H\,H^{1/2}(\Omega)$ (i.e. for any $f \in H^{1/2}(S)$) (15) is satisfied. But considering that for T ranging on V, BT spans isometrically $H'\,H^{1,2}(\Omega)$ and for $(a_j) \in R^3$, $\Sigma a_j A_j$ spans the complementary sub-space of $H'\,H^{1,2}(\Omega)$, (15) expresses essentially the decomposition of u along the two former sub-spaces: this decomposition can be achieved in a unique way by the equations

$$\text{(16)} \qquad \mathrm{BT} = (1 - \mathrm{P}_1)\, u = u - \sum_{-1}^{1}{}_j \left(\int_{\mathrm{S}_0} \mathrm{Y}_{1j}\, u \, d\mathrm{S}_0 \right) \mathrm{A}_j$$

$$\text{(17)} \qquad - \sum_{-1}^{1}{}_j\, a_j\, \mathrm{A}_j = \mathrm{P}_1\, u = \sum_{-1}{}_j \left(\int_{\mathrm{S}_0} \mathrm{Y}_{1j}\, u \, d\mathrm{S}_0 \right) \mathrm{A}_j \, .$$

The equation (16) has a unique solution T in V, by the very definition of V itself; the equation (17) is trivial since the functions A_j are linearly independent in any vector space of functions harmonic on the complement of a compact set. This theorem of existence and uniqueness seems to be general enough to meet realistic conditions as far as the simple g.b.v.p. is concerned: particularly the hypothesis $\mathrm{S} \in \mathcal{N}^{0,1}$ is very satisfactory.

However we know, via the generalized Bruns relation, that the displacement ξ between the telluroid S and the true surface of the earth Σ is a linear function of $\nabla\mathrm{T}$: e.g. for a gravimetric telluroid one has the displacement $\xi = - \mathrm{U}_0^{-1} \nabla \mathrm{T}$, where U_0 is the Marussi matrix for the normal potential.

Whence in order to have a meaningful diplacement we must be able to take the trace of $\nabla\mathrm{T}$ on S, what is not allowed if T is only in $\mathrm{H}^{1,2}(\Omega)$. We have then a problem of regularization of the solution.

REGULARIZATION OF THE SOLUTION

We aim at proving that the solution $\mathrm{T} \in \mathrm{V}$ of (14), is in fact so regular as to fulfill the relation

$$\text{(18)} \qquad \nabla \mathrm{T}_{|\mathrm{S}} \in \mathrm{L}^2(\sigma)$$

in other words we want to prove that $\partial_i \mathrm{T}|_{\mathrm{S}}$, as function of φ, λ are square integrable on the unit sphere σ.

According to formulae (10) and (13), the difficult point is to prove that

$$\text{(19)} \qquad \frac{1}{r(\alpha)} \int_{r(\alpha)}^{\mathrm{R}_0} ds\, s^2 (\partial_i u)(s, \alpha) + \frac{2\, x_i(\alpha)}{r^2(\alpha)} \int_{r(\alpha)}^{\mathrm{R}_0} ds\, s u(s, \alpha) \in \mathrm{L}^2(\sigma)\, .$$

Since on S we have $0 < r_{\min} < r < r_{\max} < \mathrm{R}_0$, we need only to prove that

$$\text{(20)} \qquad \int_\sigma d\alpha \left| \int_{r(\alpha)}^{\mathrm{R}_0} ds\, s^2 (\partial_i u)(s, \alpha) \right|^2 < +\infty \quad , \quad \int_\sigma d\alpha \left| \int_{r(\alpha)}^{\mathrm{R}_0} ds\, s u(s, \alpha) \right|^2 < +\infty\, .$$

As for the first part of (20) we have

$$\int_\sigma \mathrm{d}\alpha \left| \int_{r(\alpha)}^{R_0} \mathrm{d}s\, s^2 (\partial_i u)(s, \alpha) \right|^2 \leqq \int_\sigma \mathrm{d}\alpha \left\{ \int_{r(\alpha)}^{R_0} \mathrm{d}s\, s^2 \right\} \left\{ \int_{r(\alpha)}^{R_0} \mathrm{d}s\, s^2 (\partial_i u)(s, \alpha) \right\}^2 \leq$$

$$\leq \frac{R_0^3}{3} \int_\sigma \mathrm{d}\alpha \int_{r(\alpha)}^{R_0} \mathrm{d}s\, s^2 [(\partial_i u)(s, \alpha)]^2 \leqq \frac{R_0^3}{3} \| u \|^2_{H^{1,2}(\Omega \setminus \Omega_0)} .$$

The second term, treated similarly, gives

$$\int_\sigma \mathrm{d}\alpha \left| \int_{r(\alpha)}^{R_0} \mathrm{d}s\, s\, u(s, \alpha) \right|^2 \leqq R_0 \| u \|^2_{L^2(\Omega \setminus \Omega_0)} .$$

Accordingly one can conclude that (19) and whence (18) are true. We have thus stated a suitable theorem of existence uniqueness and regularity of the solution under fairly general conditions on the data.

The point now is: can we extend the method to the general linear Molodensky's problem? The question is callenging but difficult to be answered. In particular the method of closing the domain of the boundary operator in $H\, H^{1,2}(\Omega)$ with the graph norm leads in general to different functional spaces: this causes an essential difficulty in applying perturbative methods like in Sansò [3].

References

[1] Miranda C. (1970) – *Partial differential equations of elliptic type*. Springer, New York.

[2] Necas J. (1967) – *Les methodes directes dans la théorie des equations elliptiques*. « Editions de l'Académie Tschecoslovaque des Sciences », Prague.

[3] Sansò F. (1981) – *Recent advances in the theory of the geodetic boundary value problem*. « Reviews of Geophysics and Space Physics », Vol. 19, N. 3.