Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Antonio Lanteri, Marino Palleschi

On the adjoint system to a very ample divisor on a surface and connected inequalities. Nota I

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **71** (1981), n.5, p. 66–76.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1981_8_71_5_66_0>

Geometria algebrica. — *On the adjoint system to a very ample divisor on a surface and connected inequalities* (*). Nota I di ANTONIO LANTERI (**) e MARINO PALLESCHI (***), presentata (****) dal Corrisp. E. MARCHIONNA.

RIASSUNTO. — Siano: S una superficie algebrica proiettiva complessa non singolare, K un divisore canonico ed H un divisore molto ampio su S. Questo lavoro ha per oggetto lo studio dell'indice di autointersezione $(K + H)^2$.

Si dimostra, innanzitutto, la disuguaglianza

$$(K + H)^2 \geq 0 , \tag{I}$$

nell'ipotesi che la superficie S' ottenuta immergendo S mediante il sistema lineare completo $|H|$ non sia uno scroll. Questa disuguaglianza è connessa con alcuni risultati di Sommese e Van de Ven sulla generazione del fascio $\mathcal{O}_S(K + H)$. La dimostrazione della (I) qui fornita, evidenzia, tra le superficie per le quali la (I) vale con il segno uguale, la classe delle *rigate in coniche*.

Successivamente si osserva che, escludendo le superfici per le quali la (I) vale con il segno uguale, la disuguaglianza stessa può essere rafforzata. Si dimostra che per una superficie $S \subset \mathbf{P}^n$ con genere sezionale $g \geq 3$ che non sia nè uno scroll nè una rigata in coniche, risulta

$$(K + H)^2 \geq p_g + g - q - 2, \tag{II}$$

p_g e q essendo rispettivamente il genere geometrico e l'irregolarità di S. Si prova pure che l'uguaglianza nella (II) sussiste se e solo se S è una superficie razionale *rigata in cubiche*, con l'eccezione di due superfici razionali delle quali si descrive il modello piano.

Le disuguaglianze precenti possono essere applicate nello studio delle superfici con genere sezionale g assegnato. A titolo di esempio si classificano le superfici con $g \leq 4$ ritrovando e precisando alcuni risultati classici.

INTRODUCTION

In the last few years in connection with the problem of rebuilding a projective algebraic manifold by the knowledge of its hyperplane sections, a new interest grew out in studying the adjoint system to a very ample divisor on a surface. From this point of view the fundamental work by Sommese on the adjunction mapping [25] and Van de Ven's slick paper [27] are the most important ones. This paper is mainly concerned with the self-interesection index of a divisor adjoint to a very ample one on a surface.

(*) Lavoro eseguito nell'ambito dell'attività del G.N.S.A.G.A. del C.N.R.
(**) Istituto matematico « F. Enriques », - Via C. Saldini, 50 - 20133 Milano.
(***) Istituto Matematico - Via Università, 12 - 43100 Parma.
(****) Nella seduta del 21 novembre 1981.

Let S be an irreducible smooth complex projective algebraic surface, K and H a canonical divisor on S and a very ample one respectively, $(K+H)^2$ the self-interesection index of $K+H$ and S' the surface obtained by embedding S via the morphism associated to the complete linear system $|H|$.

Firstly (sec. 3) we prove the following fact: *If* S' *is not a scroll, then*

$$(I) \qquad (K+H)^2 \geq 0 .$$

Of course this inequality agrees with the results on the generation of the invertible sheaf $\mathcal{O}_S(K+H)$ in [25] and [27]. Our proof simply consists in studying the irreducible components of the divisors in $|K+H|$ with the advantage that it works also for smooth curves which are ample divisors on S; moreover it makes immediately evident the class of the surfaces for which equality holds in (I). We have $(K+H)^2 = 0$ *if and only if* S' *is either a Del Pezzo surface or a surface "ruled in conics".*

Then we analyze more closely the latter surfaces (sec. 4) in connection with the surfaces whose general hyperplane section is a hyperelliptic curve. This fact allows us to point out a classical result by Enriques on the surfaces with hyperelliptic hyperplane sections and to list the surfaces of $\mathbf{P}^4$ in this class.

Afterwards (sec. 5) leaving out the surfaces with $(K+H)^2 = 0$, the inequality (I) can be easily refined. *If* $S \subset \mathbf{P}^n$ *has sectional genus* $g = g(H) \geq 3$ *and it is neither a scroll nor ruled in conics, then*

$$(II) \qquad (K+H)^2 \geq p_g + g - q - 2 ,$$

where p_g *and* q *are the geometric genus and the irregularity of* S *respectively.* Moreover the surfaces for which equality holds in (II) are characterized; they are the *rational surfaces "ruled in cubics"*, with the only exceptions of two rational surfaces we explicitely describe.

Inequalities (I) and (II) apply to the projective classification of surfaces with given sectional genus g. This is a quite classical problem, more or less completely treaten for low values of g by many Authors as Picard, Castelnuovo, Enriques, Scorza and Roth. For giving an example we classify (sec. 6) the surfaces with $g \leq 4$ by means of our inequalities.

1. Background material

Here we consider only complex projective algebraic varieties. $\mathbf{P}^n$ will denote the n-dimensional complex projective space. The word surface (curve respectively) will mean projective smooth algebraic variety of dimension two (one respectively). The symbol DD' will denote the intersection index of two divisors D and D' on a surface; D^2 will be the self-intersection number of D. If C is a curve on a surface S, $D_{|C}$ will represent the divisor on C (defined mod linear equivalence $\equiv$) cut out by D.

Let S be either a surface or a curve and D a divisor on S. We will use the following standard notations:

$\mathcal{O}_S$: the structure sheaf of S;

$\mathcal{O}_S(D)$: the invertible sheaf of germs of rational functions on S which are multiples of $-D$;

$H^q(S, \mathcal{O}_S(D))$: the q-th cohomology complex vector space of S with coefficients in $\mathcal{O}_S(D)$;

$h^q(D) = h^q(\mathcal{O}_S(D)) = \dim_{\mathbf{C}} H^q(S, \mathcal{O}_S(D))$;

$\chi(\mathcal{O}_S) = \sum_{q=0}^{\dim S} (-1)^q h^q(\mathcal{O}_S)$;

$|D|$: the complete linear system defined by D;

Φ_D : the rational map $S \dashrightarrow \mathbf{P}^n$ $(n = h^0(D) - 1)$ defined by $|D|$, when $n \geq 0$;

K or K_S : a canonical divisor on S;

$|D| \cdot C$: the linear series which the complete linear system $|D|$ on a surface S cuts out on a curve $C \subset S$;

$p_g = p_g(S)$: the geometric genus of a surface S;

$q = q(S)$: its irregularity;

$g(C)$: the genus of a curve C.

A surface S is *ruled* if it is birational to the product $B \times \mathbf{P}^1$ of a curve B and the projective line; S is *rational* if it is birational to $\mathbf{P}^2$. By writing $S \subset \mathbf{P}^n$ we mean that S is not contained in any hyperplane. If the linear system cut out on S by the hyperplanes of $\mathbf{P}^n$ is complete we say that S is *linearly normal.*

From now on S will be a surface, H a general element of a very ample complete linear system $|H|$, $d = H^2$ and $g = g(H)$. Sometimes we will identify S with the surface $S' = \Phi_H(S)$: in that case d can be thought as the degree of S' and H as a general hyperplane section of S'. If $S \subset \mathbf{P}^n$, H will always denote a general hyperplane section of S; $g = g(H)$ will be called *sectional genus* of S.

Consider the exact residue sequence

$$0 \to \mathcal{O}_S(K) \to \mathcal{O}_S(K + H) \to \mathcal{O}_H(K_H) \to 0. \tag{1.1}$$

By Kodaira vanishing theorem, (1.1) induces the exact sequence

$$\begin{aligned} 0 \to H^0(S, \mathcal{O}_S(K)) \to H^0(S, \mathcal{O}_S(K + H)) \xrightarrow{\alpha} \\ \xrightarrow{\alpha} H^0(H, \mathcal{O}_H(K_H)) \xrightarrow{\beta} H^1(S, \mathcal{O}_S(K)) \to 0, \end{aligned} \tag{1.2}$$

involving the formula

$$h^0(K + H) = p_g + g - q. \tag{1.3}$$

Remark 1.1. One has $g \geq q$; moreover if $g = q$ then $p_g = 0$. In fact the homomorphism β in (1.2) is surjective and so $g \geq q$. Suppose $g = q$; then β is also injective; therefore $\mathrm{Im}\, \alpha = 0$ i.e. $|K + H| \cdot H = \emptyset$. As H is very ample (really since H is ample and $h^0(H) \geq 2$), $|K + H| = \emptyset$ too and by (1.3) we are done.

Notice that Remark 1.1 and (1.3) force the following trivial fact.

Remark 1.2. If $h^0(K + H) \leq 1$ and $g \neq q$, then $p_g = 0$ and $g = q + 1$.

Remark 1.3. Suppose $q = 0$ and $g \geq 2$; then the complete linear system $|K + H|$ has no fixed components and is base point free. In particular the rational map $\Phi_{K+H} : S \dashrightarrow \mathbf{P}^n$ is a morphism (1). By absurd let p be either a base point of $|K + H|$ or a point lying on a fixed component of its and let $\tilde{H}$ be a general element of $|H|$ through p. In this case p is a fixed point of the series $|K + H| \cdot \tilde{H}$. As $q = 0$, the exact sequence (1.2) becomes

$$0 \to H^0(S, \mathcal{O}_S(K)) \to H^0(S, \mathcal{O}_S(K + H)) \xrightarrow{\alpha} H^0(\tilde{H}, \mathcal{O}_{\tilde{H}}(K_{\tilde{H}})) \to 0,$$

hence $|K + H| \cdot \tilde{H} = |K_{\tilde{H}}|$. But it is well known that $|K_{\tilde{H}}|$ is base point free.

Now consider the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(H) \to \mathcal{O}_H(H|_H) \to 0$$

and the induced exact cohomology sequence

$$\begin{aligned} (1.4) \qquad & 0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(H)) \to H^0(H, \mathcal{O}_H(H|_H)) \to \\ & \to H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{O}_S(H)) \to H^1(H, \mathcal{O}_H(H|_H)) \to \cdots . \end{aligned}$$

Remark 1.4. If $d > 2g - 2$ then S is a ruled surface; moreover $h^1(H) \leq q$.

Indeed genus formula gives $HK = 2g - 2 - d < 0$ and then $P_n(S) = h^0(nK) = 0$ for any integer $n \geq 1$; hence S is a ruled surface (e.g. see [1], p. 112). Moreover $H|_H$ is a non-special divisor so (1.4) implies $h^1(H) \leq q$.

By *geometrically ruled* surface we mean a surface S endowed with a morphism $\pi : S \to B$ on a curve B such that $\pi^{-1}(b) \simeq \mathbf{P}^1$ for any $b \in B$. By Noether-Enriques theorem (see [1], p. 35) such a surface is a ruled one. Notice also that $q = g(B)$.

By the theory of minimal models and by the structure theorem of birational morphisms (see [24], pp. 85–100, [11], pp. 411–412] the following facts are known: any ruled surface S—not isomorphic to $\mathbf{P}^2$—dominates a suitable geometrically ruled surface S_0 by a morphism $\eta : S \to S_0$. Let s be the number of blowings-up by means of which η factorizes. One has

$$K_S^2 = K_{S_0}^2 - s \quad , \quad K_{S_0}^2 = 8(1 - q) \quad , \quad K_{\mathbf{P}^2}^2 = 9 .$$

All these facts can be summarized in the following.

(1) More generally this result holds if $g \neq q$ (see [25], p. 387).

Remark 1.5. Let S be a ruled surface. If $S \not\cong \mathbf{P}^2$, then $K_S^2 \leq 8(1-q)$, equality holding if and only if S is a geometrically ruled surface.

Now let S_0 be a geometrically ruled surface over B and denote by C_0 a fundamental section, i.e. a section of minimal self-intersection number $-e = C_0^2$. The integer e is named the *invariant* of S_0. Call F a fibre of π. Then for any divisor D on S_0 one has (see [11], p. 370) $D \equiv aC_0 + bF$ $(a, b \in \mathbf{Z})$, where $\equiv$ denotes numerical equivalence. In particular (see [11], p. 373).

$$K \equiv -2\,C_0 + (2\,q - 2 - e)\,F\,. \tag{1.5}$$

Now let $X \subset \mathbf{P}^n$ be a geometrically ruled surface over B. If any fibre F of X is a line, we say that X is a *scroll* over B. An easy computation shows that a geometrically ruled surface $X \subset \mathbf{P}^n$ is a scroll if and only if its hyperplane divisor has the form

$$H \equiv C_0 + mF \qquad (m \in \mathbf{Z})\,. \tag{1.6}$$

Let $Y \subset \mathbf{P}^n$ be a surface; if Y is not a scroll and if there exists a morphism $h : Y \to B$ over a curve B whose fibres are conics we say that Y is *ruled in conics*. By Noether-Enriques theorem (see [1], p. 35) such a Y is ruled. Moreover, Y being irreducible, it admits only a finite number δ of singular fibres and each one of them consists of two intersecting lines. If $\delta = 0$, Y is a geometrically ruled surface. In this case we say that Y is *geometrically ruled in conics* (2).

Let $Y \subset \mathbf{P}^n$ be a geometrically ruled surface. It is straightforward to verify that Y is geometrically ruled in conics if and only if its hyperplane section has the form

$$H \equiv 2\,C_0 + mF \qquad (m \in \mathbf{Z})\,. \tag{1.7}$$

2. Preliminary lemmata

Consider a surface S and a general element H of a very ample linear system $|H|$ on S; put $g = g(H)$.

When $|K + H| \neq \varnothing$, any divisor $D \in |K + H|$ can be represented in the form

$$D = \sum_{i=1}^{r} C_i$$

(2) Notice that the quartic surface $\Sigma = \Phi_H(\mathbf{P}^1 \times \mathbf{P}^1) \subset \mathbf{P}^5$ where $H \equiv C_0 + 2\,F$ admits the fibration in conics corresponding to the pencil $|C_0|$. Nevertheless Σ is not ruled in conics, being a scroll. On the other hand it can be seen that Σ is the unique geometrically ruled surface with this property.

where the C_i's are the irreducible (possibly repeated) components of D Obviously

$$D^2 = (K + H)^2 = \sum_{i=1}^{r} C_i^2 + \sum_{j \neq i} C_i C_j = \tag{2.1}$$

$$= \sum_{i=1}^{r} \left(C_i^2 + C_i \sum_{j \neq i} C_j \right) = \sum_{i=1}^{r} C_i D .$$

LEMMA 2.1. *Suppose either* 1) $h^0(K + H) \geq 2$ *or* 2) $h^0(K + H) = 1$ *and the divisor* $D \in |K + H|$ *has two distinct irreducible components at least. Then*

(a) *every addendum* $C_i D$ *in* (2.1) *is non-negative*;

(b) $(K + H)^2 \geq 0$.

Proof. Of course it is enough to prove (a). Suppose $r = 1$, i.e. $D = C_1$; as D varies in a pencil at least by assumption, one has $D^2 = DC_1 \geq 0$ obviously. Now suppose $r > 1$. If $C_i^2 \geq 0$ then (a) holds trivially for the addendum

$$C_i D = C_i^2 + C_i \sum_{j \neq i} C_j .$$

On the contrary suppose $C_i^2 < 0$; we have $C_i D = C_i (K + H) \geq C_i K + 1$ and by genus formula $C_i K = 2g(C_i) - 2 - C_i^2 > -2$. This shows that (a) holds in this case too.

LEMMA 2.2. *Consider an effective divisor* $L = \sum_{i=1}^{m} r_i L_i$ (L_i*'s being the irreducible components*) *on* S *such that* $L_i^2 = -1$, $L_i H = 1$ $(i = 1, \cdots, m)$ *and* $L_i \neq L_j$ *for* $i \neq j$. *Suppose* $L_i L = 0$ *for any i. Then for each i there exists an index* i' $(1 \leq i' \leq m)$ *such that* $L_i L_{i'} = 1$, $L_i L_h = L_{i'} L_h = 0$ *for any* $h \neq i, i'$ *and* $r_i = r_{i'}$; *in particular m is even.*

Proof. Our assumptions imply $0 \leq L_i L_j \leq 1$ for $i \neq j$, as Φ_H transforms each L_i in a line; moreover in view of the equality

$$0 = L_i L = L_i \left(r_i L_i + \sum_{j \neq i} r_j L_j \right),$$

there exist p components $L_{j_1}, \cdots, L_{j_p}$ of L such that

$$r_i = r_{j_1} + \cdots + r_{j_p}; \tag{2.2}$$

$$L_i L_{j_1} = \cdots = L_i L_{j_p} = 1 . \tag{2.3}$$

Obviously for any other L_h $(h \neq j_1, \cdots, j_p)$ one has $L_h L_i = 0$. Consider the component L_{j_1} and the equalities

$$L_i L = L_i \left(r_i L_i + \sum_{j \neq i} r_j L_j \right) = -r_i + r_{j_1} L_i L_{j_1} + \varepsilon = 0 ,$$

$$L_{j_1} L = L_{j_1} \left(r_{j_1} L_{j_1} + \sum_{j \neq j_1} r_j L_j \right) = -r_{j_1} + r_i L_{j_1} L_i + \varepsilon' = 0 ,$$

where ε and ε' are non-negative integers. There follows $r_{j_1} L_i L_{j_1} \leq r_i$ and $r_i L_i L_{j_1} \leq r_{j_1}$; so (2.3) implies $r_i = r_{j_1}$ and, by (2.2), $p = 1$. Now put $i' = j_1$; to conclude it is enough to show that $L_{i'} L_h = 0$ for $h \neq i, i'$, but this is immediate in view of the symmetry between the indexes i and i'.

LEMMA 2.3. *Suppose* $h^0(K + H) \geq 2$ *and* $(K + H)^2 = 0$. *Then any divisor* $D \in |K + H|$ *can be expressed by means of its connected components* F_i*'s* $(i = 1, \cdots, s)$ *in the form*

$$D = \sum_{i=1}^{s} r_i F_i$$

with $F_i H = 2$.

Proof. By Lemma 2.1, (a), the assumption $(K + H)^2 = 0$ implies

$$C_i D = C_i K + C_i H = C_i^2 + C_i \sum_{j \neq i} C_j = 0 \tag{2.4}$$

for any $i = 1, \cdots, s$. Therefore $C_i^2 \leq 0$ and $C_i K = - C_i H \leq -1$ and then

$$-2 \leq 2g(C_i) - 2 = C_i^2 + C_i K \leq -1 . \tag{2.5}$$

On the other hand one concludes that $C_i^2 + C_i K = -2$, since $C_i^2 + C_i K$ is even. As $C_i K \leq -1$, there are only two possibilities:

$$C_i^2 = 0 \qquad \text{and} \qquad C_i K = -2 \; (= - C_i H) ; \tag{2.6}$$

$$C_i^2 = -1 \qquad \text{and} \qquad C_i K = -1 \; (= - C_i H) . \tag{2.7}$$

For any component C_i verifying (2.6) we have $C_i C_j = 0$ for any j, by (2.4). Therefore $F_i = C_i$ is a connected component of D and $F_i H = C_i H = - C_i K = 2$. So if all the C_i's fulfill (2.6), the Lemma is proven. Otherwise if there exists a C_i verifying (2.7), consider the divisor L obtained from D by deleting its (possible) components satisfying (2.6) and rename L_i's the remaining components C_i's. In view of (2.4) it is easy to see that $L_i L = 0$. By applying Lemma 2.2 we see that $F = L_i + L_{i'} = C_i + C_{i'}$ is a connected component of L, hence of D, and $FH = 2$, by (2.7).

For the sequel we need to know what happens when $h^0(K + H) \leq 1$. In sec. 3 we will analize the case $g = q$; now we are going to show

LEMMA 2.4. *Suppose* $g \neq q$ *and* $h^0(K + H) \leq 1$. *Then either*

a) $\Phi_H(S)$ *is a Del Pezzo surface, or*

b) S *is a ruled surface with* $q = 2$ *and its fibre* F *verifies* $FH = 2$.

Proof. By means of Remark 1.2, our assumptions imply

$$p_g = 0 \tag{2.8}$$

and

$$g = q + 1 . \tag{2.9}$$

If $q = 0$, Remark 1.4 and (1.4) show that $\Phi_H(S)$ is a rational surface of degree d in $\mathbf{P}^d$, hence case a) occurs. Now suppose $q \geq 1$; in view of (2.8) the image of the Albanese map $\alpha : S \to \mathrm{Alb}(S)$ is a curve B of genus q (see [1], pp. 85–86). Since H cannot be a fibre of α, the morphism $\alpha|_H : H \to B$ is surjective; let m be its degree. As $g \neq q$, one has $m \geq 2$. By Riemann-Hurwitz formula the total branching order of $\alpha|_H$ is $r = 2q(1-m) + 2m$. So, as $r \geq 0$, one deduces

$$q \leq \frac{m}{m-1}, \tag{2.10}$$

which means $q \leq 2$. Thus $g = 2$ or $g = 3$, by (2.9). In both cases $d = H^2 > 2g - 2$ (3) and S is ruled by Remark 1.4. On the other hand it cannot be $q = 1$; indeed, for a smooth curve H of genus g contained in an unrational ruled surface, which is not a section of a geometrically ruled surface, it must be $H^2 \leq 4g - 4$ (see [12], Corollary 2.4).

3. The first inequality and a characterization of the surfaces ruled in conics

In this sec. we study the self-intersection number $(K + H)^2$. We start with a rather classical characterization of the surfaces with $g = q$. First of all we have the following

Remark 3.1. There results $(K + 2H)^2 \geq 0$. It is enough to prove this inequality when $h^0(K + 2H) \leq 1$, in view of Lemma 2.1. To do this consider the exact sequence

$$0 \to \mathcal{O}_S(K + H) \to \mathcal{O}_S(K + 2H) \to \mathcal{O}_H(K_H + H|_H) \to 0\,;$$

by Kodaira vanishing theorem one gets $h^0(K + 2H) = h^0(K + H) + h^0(K_H + H|_H)$. This formula expresses $h^0(K + 2H)$ as a sum of non-negative integers; so the inequality $h^0(K + 2H) \leq 1$ involves $h^0(K_H + H|_H) \leq 1$ and by Riemann-Roch theorem, this reads $g - 1 + H^2 \leq 1$. There follows $H^2 \leq 2$ i.e. $\Phi_H(S)$ is either $\mathbf{P}^2$ or the quadric surface and our inequality holds trivially in both cases.

PROPOSITION 3.1. *Equality $g = q$ holds if and only if $\Phi_H(S)$ is either $\mathbf{P}^2$, the Veronese surface, or a scroll.*

(3) Identify S with $\Phi_H(S)$ and think of H as a smooth hyperplane section of S. We have $d = H^2 \geq 4$, since $g \geq 2$, and, if $d = 4$, H is a plane curve of genus three. In this case S is a quartic surface in $\mathbf{P}^3$ and then it is regular. So, as we are dealing with irregular surfaces, it is $d > 4$.

Proof. For each surface $\Phi_H(S)$ listed before one has $g = q$ trivially. Conversely, suppose $g = q$; then Remark 3.1 and genus formula give

$$0 \leq (K + 2H)^2 = K^2 - 8(1 - q). \tag{3.1}$$

We continue the proof in two steps.

Step 1. S *is ruled.* Indeed, if $g = 0$, S is rational by Noether-Enriques theorem; if $g = 1$, since $d = H^2 > 0 = 2g - 2$, S is ruled by Remark 1.4. Finally suppose $g = q \geq 2$; then Remark 1.1 involves $p_g = 0$. Taking also into account (3.1) and Noether formula ([9], p. 601), the topological Euler-Poincaré characteristic of S is

$$\chi(S) = 12\,\chi(\mathcal{O}_S) - K^2 \leq 4(1 - q) < 0;$$

so S is ruled by a classical theorem due to Castelnuovo and De Franchis (see [2], p. 213).

Step 2. Suppose $S \simeq \mathbf{P}^2$; then $H \equiv kL$, L being the effective generator of the Picard group of S. As $q = 0$, condition $g = q$ implies either $k = 1$ or $k = 2$. In the former case $\Phi_H(S) = \mathbf{P}^2$ whilst in the latter one $\Phi_H(S)$ is the Veronese surface. Now suppose S is not isomorphic to $\mathbf{P}^2$; then by Remark 1.5 and formula (3.1) one gets $K^2 = 8(1 - q)$. Once again Remark 1.5 shows S is a geometrically ruled surface. In this case H can be written as $H \equiv aC_0 + bF$ (see sec. 1). Recalling (1.5), genus formula supplies

$$2g - 2 = H^2 + HK = (a - 1)(2b - ae) + 2a(q - 1). \tag{3.2}$$

Now $\Phi_H(S)$ is a scroll if and only if (1.6) holds, i.e. if and only if $a = 1$. Suppose $a \neq 1$; then, as $g = q$, (3.2) involves $q = 0$, $e = 0$ and $b = 1$, in view of the ampleness conditions on H (see [11], pp. 380–382). But in view of the symmetry between C_0 and F, when $q = e = 0$, $\Phi_H(S)$ is a scroll in this case too.

Now it is possible to prove

THEOREM 3.1. *Let* S *be a surface*, H *a very ample divisor on* S *and suppose* $\Phi_H(S)$ *is not a scroll*; *then*

$$(K + H)^2 \geq 0, \tag{3.3}$$

equality holding if and only if $\Phi_H(S)$ *is either a Del Pezzo surface or ruled in conics.*

Proof. Let us start by proving (3.3). If $h^0(K + H) \geq 2$, Lemma 2.1, (b) shows (3.3). Suppose $h^0(K + H) \leq 1$; if $g = q$ our assumption and Proposition 3.1 imply $\Phi_H(S)$ is either $\mathbf{P}^2$ or the Veronese surface and such surfaces fulfill (3.3) obviously. So we have only to consider the case $g \neq q$ when

$h^0(K + H) \leq 1$. By Remark 1.2 one has $p_g = 0$ and $g = q + 1$; so $h^0(K + H) = 1$ by (1.3). There are three cases:

i) $K + H \equiv 0$; thus equality holds trivially in (3.3);

ii) the effective divisor $D \in |K + H|$ has two distinct irreducible components at least; then (3.3) holds by Lemma 2.1;

iii) the effective divisor $D \in |K + H|$ can written as $D = nC$ $(n \geq 1)$, C being irreducible; in this case we have $C(K + H) = CD = nC^2$, so $CK = nC^2 - CH$, and genus formula gives $-2 \leq 2g(C) - 2 = C^2 + CK = (n + 1)C^2 - CH$. Since H is very ample the previous inequality involves $(n + 1)C^2 \geq -1$. But $(n + 1)C^2 \neq -1$ of course. Then $C^2 \geq 0$ and so

$$(K + H)^2 = D^2 = n^2 C^2 \geq 0 .$$

Now we are going to characterize equality in (3.3). If $\Phi_H(S)$ is a Del Pezzo surface then $K + H \equiv 0$ (see [11], p. 401) and then equality holds in (3.3) as previously stated in *i*). Suppose $\Phi_H(S)$ is ruled in conics; then S admits a fibration whose general fibre Γ verifies $\Gamma H = 2$. Then there exists a morphism $\eta : S \to S_0$ where S_0 is a geometrically ruled surface and η factorizes by means of δ blowings-up σ_i's with centers p_i's belonging to distinct fibres of S_0. Let $E_i \subset S$ be the exceptional curve corresponding to p_i $(i = 1, \cdots, \delta)$. Obviously $E_i H = 1$ $(i = 1, \cdots, \delta)$ since $\Gamma H = 2$, and $C = \eta(H)$ is a two-secant curve in S_0; this means that $C \equiv 2C_0 + bF$. Recalling also (1.5) one deduces

$$(K_{S_0} + C)^2 = 0 . \tag{3.4}$$

Moreover $|H|$ corresponds to the linear system $|C - p_1 - \cdots - p_\delta|$ on S_0 (in the sense of [11], pp. 395–396) and $\eta^* C \equiv H + E_1 + \cdots + E_\delta$. As $K \equiv \eta^* K_{S_0} + E_1 + \cdots + E_\delta$, one gets immediately $K + H \equiv \eta^*(K_{S_0} + C)$ and then (3.4) implies $(K + H)^2 = 0$. Conversely suppose $(K + H)^2 = 0$. Then, if $h^0(K + H) \geq 2$, the rational map Φ_{K+H} is a morphism over a curve B. Indeed call Z and M the fixed and the moving part of $|K + H|$ respectively. Then $K + H \equiv Z + M$ and $(K + H)^2 = (K + H)Z + ZM + M^2$. Of course $ZM \geq 0$ and $M^2 \geq 0$; moreover $(K + H)Z \geq 0$, by Lemma 2.1, *a*). So $(K + H)^2 \geq M^2$. As $(K + H)^2 = 0$, we have $M^2 = 0$ and then $\Phi_{K+H} = \Phi_M$ is a morphism over a curve B. Consider now the following Stein factorization

$$\begin{tikzcd} S \arrow[rr, "\Phi_{K+H}"] \arrow[dr, "\pi"'] & & B \\ & \tilde{B} \arrow[ur] & \end{tikzcd}$$

The general fibre F of π is a connected component of a divisor $D \in |K + H|$. Hence, by Lemma 2.3, the morphism $\pi : S \to \tilde{B}$ exhibits S as a ruled surface over $\tilde{B}$ with $FH = 2$. Finally, if $h^0(K + H) \leq 1$, as we supposed

$(K+H)^2 = 0$, $\Phi_H(S)$ can be neither $\mathbf{P}^2$ nor the Veronese surface, and so, in view of Proposition 3.1 and Lemma 2.4 we are done.

Note that if we simply suppose H *is a (smooth) curve in* S *which is an ample divisor and* $h^0(H) \geq 2$, the most part of our results continue to hold, within obvious modifications. In particular, Proposition 3.1 becomes: *equality* $g = q$ *holds if and only if either* $S \simeq \mathbf{P}^2$ *and* $\mathcal{O}_S(H) = \mathcal{O}_{\mathbf{P}^2}(n)$, $(n = 1, 2)$, *or* S *is a geometrically ruled surface and* H *is a section.* All lemmata in sec. 2 continue to hold unless Lemma 2.4. Nevertheless the same argument there used proves the following. *Assume* $g \neq q$, $H^2 \geq 5$ *and* $h^0(K+H) \leq 1$. *Then either* S *is a Del Pezzo surface and* $K \equiv -H$ *or* S *is a ruled surface with* $q = 2$ *and* H *is a 2-section.*

Using these facts, Theorem 3.1 can be restated in a more general form.

THEOREM 3.2. *Let* S *be a surface and* H *a (smooth) curve in* S *which is an ample divisor and such that* $h^0(H) \geq 2$. *If* S *is not a geometrically ruled surface having* H *as a section, then* $(K+H)^2 \geq 0$. *If further, either* $H^2 \geq 5$ *or* H *is very ample, then equality holds if and only if either* S *is a Del Pezzo surface and* $H \equiv -K$ *or* S *is a ruled surface and* H *is a 2-section.*

Whenever we apply the previous results to the hyperplane sections of an embedded surface Theorem 3.1 has the following.

COROLLARY 3.1. *Let* $X \subset \mathbf{P}^N$ *be a surface of degree* d *and sectional genus* g. *Then*

I) X *is a scroll if and only if* $K_X^2 = 8(1-g)$;

II) *if* X *is not a scroll one has*

$$d \leq 4g - 4 + K_X^2, \tag{3.5}$$

and equality holds if and only if X *is either a Del Pezzo surface, a projection of its or ruled in conics.*

Remark 3.2. Let $X \subset \mathbf{P}^N$ be a surface of degree d and sectional genus g. Suppose X is ruled in conics; then the ruling has

$$\delta = 8(1-q) + 4(g-1) - d$$

singular fibres. Suppose $\pi : X \to B$ is the morphism exhibiting X as ruled in conics; obviously $g(B) = q$. If F is a general fibre of π and F_i $(i = 1, \cdots, \delta)$ is a singular one, then the Euler-Poincaré characteristic of X is (see [1], p. 156).

$$\chi(X) = \chi(B)\,\chi(F) + \sum_{i=1}^{\delta} (\chi(F_i) - \chi(F)),$$

i.e. $\chi(X) = 4(1-q) + \delta$. On the other hand Noether formula and Corollary 3.1 give $\chi(X) = 12\,\chi(\mathcal{O}_X) - K_X^2 = 12(1-q) + 4(g-1) - d$.

The references of this Nota I can be found at the end of the same titled Nota II which is integral part of this work.