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The bounce problem, on n-dimensional Riemannian manifolds

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Analisi matematica. — *The bounce problem on* n*-dimensional Riemannian manifolds*. Nota di GIUSEPPE BUTTAZZO e DANILO PERCIVALLE, presentata (*) dal Corrisp. E. DE GIORGI.

RIASSUNTO. — In questo lavoro vengono generalizzati i risultati relativi al problema del rimbalzo unidimensionale studiato in [5]. Precisamente si considera un punto mobile su una varietà Riemanniana V n-dimensionale, soggetto all'azione di un potenziale variabile nel tempo e vincolato a restare in una parte W di V avente un bordo di classe C^3 contro cui il punto « rimbalza ».

Lo studio del problema richiede l'uso di metodi di Γ-convergenza del tipo usato in [5], metodi che sembrano caratteristici per lo studio di problemi in cui può mancare l'unicità della soluzione o la sua dipendenza continua dai dati.

Let V be an n-dimensional Riemannian manifold without boundary and of class C^3; for every $q \in V$ we denote by $\langle\, ,\, \rangle_q$ the scalar product on the tangent space $T_q V$; we assume that the metric tensor field is of class C^2.

If $g : V \to \mathbf{R}$ is a differentiable function, we denote by ∇g the gradient of g defined by

$$\langle \nabla g(q), v\rangle_q = dg(q)(v) \qquad \text{for every} \quad v \in T_q V .$$

Let $f : V \to \mathbf{R}$ be a function of class C^3; we suppose that $df(q) \neq 0$ for every $q \in V$ such that $f(q) = 0$.

We study the bounce problem for a material point, whose position at time t will be indicated by $q(t)$. This point is subjected to a potential $U(t, q)$ and moves in the region $W = \{q \in V : f(q) \geq 0\}$ bouncing against the submanifold $S = \{q \in V : f(q) = 0\}$.

Given $T > 0$ we denote by Lip the space of Lipschitz functions from $[0, T]$ into V endowed with the L^∞-topology and by L^1 the space $L^1(0, T; C^2(V))$ with its usual topology.

(*) Nella seduta del 26 giugno 1981.

We say that a pair $(U, q) \in L^1 \times \mathrm{Lip}$ solves the bounce problem $(\mathscr{P})$ (or that q is a solution of the bounce problem $(\mathscr{P})$ with potential U) if

$$(\mathscr{P}) \quad \begin{cases} i) & f(q(t)) \geq 0 \text{ for every } t \in [0, T] \\ ii) & \text{there exists a positive measure } \mu \text{ on } (0, T) \text{ such that } q \text{ is an extremal for the functional} \\ & F(v) = \int_0^T [\tfrac{1}{2} \langle v'(t), v'(t)\rangle + U(t, v(t))]\, dt + \int_0^T f(v(t))\, d\mu \\ & \text{and } spt\, \mu \subseteq \{t \in [0, T] : f(q(t)) = 0\} \\ iii) & \text{for every } t_1, t_2 \in [0, T] \text{ the energy-relation holds:} \\ & \langle q'_+, q'_+\rangle \big|_{t_1}^{t_2} = \langle q'_-, q'_-\rangle \big|_{t_1}^{t_2} = 2 \int_{t_1}^{t_2} \langle \nabla U(t, q(t)), q'(t)\rangle\, dt \end{cases}$$

where q'_+ and q'_- respectively denote the right and left derivatives of q, whose existence is guaranteed by *ii*). By *iii*) the functions $\langle q'_+, q'_+\rangle$ and $\langle q'_-, q'_-\rangle$ coincide; their common value will be indicated by $\langle q', q'\rangle$.

We consider now a sequence of penalizing functions ψ_h satisfying the following properties:

i) ψ_h is continuous, $\psi_h \geq 0$, $\psi_h(x) = 0$ if $x \geq 0$

ii) $\psi_h \to +\infty$ uniformly on compact subsets of $(-\infty, 0)$

iii) $\displaystyle\lim_{\substack{h \to +\infty \\ x \to 0^-}} \frac{\psi_h(x)}{\alpha_h(x)} = +\infty$ where $\displaystyle\alpha_h(x) = \int_x^0 \psi_h(z)\, dz$.

We say that a pair $(U, q) \in L^1 \times \mathrm{Lip}$ solves the penalized problem $(\mathscr{P}_h)$ (or that q is a solution of $(\mathscr{P}_h)$ with potential U) if q is an extremal for the functional

$$F_h(v) = \int_0^T [\tfrac{1}{2} \langle v'(t), v'(t)\rangle + U(t, v(t)) - \alpha_h(f(v(t)))]\, dt .$$

Remark. It is easy to see that if (U, q) satisfies $(\mathscr{P})$ *ii*) then, in local coordinates we have (the summation convention is adopted):

$$\frac{d}{dt}(a_{ij}(q)\, q'_j) = \frac{\partial}{\partial q_i}[\tfrac{1}{2} a_{rs}(q)\, q'_r\, q'_s + U(t, q)] + \mu \frac{\partial f}{\partial q_i}(q)$$

where a_{ij} denote the coefficients of the metric tensor.

Similarly, if (U, q) solves the problem $(\mathscr{P}_h)$ then, in local coordinates we have

$$\frac{d}{dt}(a_{ij}(q) q_j') = \frac{\partial}{\partial q_i}[\tfrac{1}{2} a_{rs}(q) q_r' q_s' + U(t, q) - \alpha_h(f(q))] .$$

We are interested in the Cauchy problems for $(\mathscr{P}_h)$ and $(\mathscr{P})$; nevertheless we cannot assign the usual Cauchy data since they are not stable as $h \to +\infty$. In fact, if $U \in L^1$ is a potential, q_h solves $(\mathscr{P}_h)$ with potential U, q solves $(\mathscr{P})$ with potential U and $q_h \to q$ uniformly, then the usual Cauchy data of the problem $(\mathscr{P}_h)$ may be not convergent to the Cauchy data of the problem $(\mathscr{P})$. Moreover the usual Cauchy data for the problem $(\mathscr{P})$ are discontinuous as functions of time t. These facts lead us to introduce a new kind of "initial trace" for $(\mathscr{P}_h)$ and $(\mathscr{P})$, which is more stable than the usual one as $h \to +\infty$.

Let

$$E_h = \{(U, q) \in L^1 \times \mathrm{Lip} : (U, q) \text{ solves } (\mathscr{P}_h)\}$$

$$E = \{(U, q) \in L^1 \times \mathrm{Lip} : (U, q) \text{ solves } (\mathscr{P})\}$$

$$Y_h = \{q \in \mathrm{Lip} : \exists U \in L^1 \text{ with } (U, q) \in E_h\}$$

$$Y = \{q \in \mathrm{Lip} : \exists U \in L^1 \text{ with } (U, q) \in E\} ;$$

setting $\mathscr{B} = \mathbf{R} \times V \times TV \times TV \times TV$ we define the "initial traces"

$$\mathscr{T}_h : [0, T] \times Y_h \to \mathscr{B} \quad \text{and} \quad \mathscr{T} : [0, T] \times Y \to \mathscr{B} \quad \text{by}$$

$$\mathscr{T}(t, q) = \left(e_h(q)(t), q(t), q_\tau'(t), f(q(t)) q'(t), \frac{1}{h} q'(t)\right)$$

$$\mathscr{T}(t, q) = (e(q)(t), q(t), q(t), q_\tau'(t), f(q(t)) q'(t), 0)$$

where q_τ' denotes the vector $\langle \nabla f(q), \nabla f(q)\rangle q' - \langle q', \nabla f(q)\rangle \nabla f(q)$, and $e_h(q)$, $e(q)$ are respectively the energies $\frac{1}{2}\langle q', q'\rangle + \alpha_h(f(q))$ and $\frac{1}{2}\langle q', q'\rangle$.

It is easy to verify that $\mathscr{T}_h$, $\mathscr{T}$ are continuous with respect to t for every fixed q in Y_h, Y respectively.

We remark that, if $f(q(t_0)) > 0$, then to assign $\mathscr{T}(t_0, q)$ is equivalent to assigning the Cauchy data $q(t_0)$, $q'(t_0)$ for the problem $(\mathscr{P})$; to assign $\mathscr{T}_h(t_0, q)$ is, for the problem $(\mathscr{P}_h)$, always equivalent to assigning the usual Cauchy data.

We set for every $t \in [0, T]$

$$\mathscr{A}_h(t) = \{(b, U, q) \in \mathscr{B} \times L^1 \times \mathrm{Lip} : (U, q) \in E_h, \mathscr{T}_h(t, q) = b\}$$

$$\mathscr{A}(t) = \{(b, U, q) \in \mathscr{B} \times L^1 \times \mathrm{Lip} : (Y, q) \in E, \mathscr{T}(t, q) = b\} ;$$

we notice that, for fixed $t \in [0, T]$, the relation $\mathscr{A}(t)$ does not uniquely characterize q as a function of (b, U) (see [6]). Failing uniqueness for the problem $(\mathscr{P})$ with fixed initial conditions, the following question naturally arises:

may every solution of the problem $(\mathscr{P})$ be obtained as a limit of solutions of problems $(\mathscr{P}_h)$?

We answer this question in the affirmative by allowing a suitable "mobility" of potential and of initial conditions. To do this we use the notions of Γ-limits.

If X is a set and $M \subseteq X$ we define

$$\delta_M(x) = \begin{cases} 0 & \text{if } x \in M \\ +\infty & \text{otherwise;} \end{cases}$$

we recall (see [4], [8], [9]) that we say

$$\delta_{\mathscr{A}(t)} = \Gamma(\mathbf{N}, \mathscr{B}^-, (L^1)^-, (L^\infty)^-) \lim_h \delta_{\mathscr{A}_h(t)}$$

if and only if the following properties hold:

I) If $b_h \to b$ in $\mathscr{B}$, $U_h \to U$ in L^1, $q_h \to q$ in L^∞ with $(b_h, U_h, q_h) \in \mathscr{A}_h(t)$ for infinitely many $h \in \mathbf{N}$, then $(b, U, q) \in \mathscr{A}(t)$.

II) If $(b, U, q) \in \mathscr{A}(t)$, then there exist $b_h \to b$ in $\mathscr{B}$, $U_h \to U$ in L^1, $q_h \to q$ in L^∞ such that $(b_h, U_h, q_h) \in \mathscr{A}_h(t)$ for all h large enough.

Our purpose is to obtain the following result:

THEOREM 1. *For every* $t \in [0, T]$ *we have*

$$\delta_{\mathscr{A}(t)} = \Gamma(\mathbf{N}, \mathscr{B}^-, (L^1)^-, (L^\infty)^-) \lim_h \delta_{\mathscr{A}_h(t)} .$$

The proof of the Theorem 1 mainly consists in the following steps:

PROPOSITION 2. *For every* $t \in [0, T]$

$$\delta_{\mathscr{A}(t)} \leq \Gamma(\mathbf{N}^-, \mathscr{B}^-, (L^1)^-, (L^\infty)^-) \lim_h \delta_{\mathscr{A}_h(t)} .$$

This Proposition proves property I). In order to prove property II), we need the notion of regular solution for the problem $(\mathscr{P})$;

DEFINITION 3. *We set for every* $t \in [0, T]$

$$D(t) = \{(b, U, q) \in \mathscr{A}(t) : \forall \tilde{q} \in \text{Lip}, (b, U, \tilde{q}) \in \mathscr{A}(t) \Rightarrow q = \tilde{q}\} .$$

We call $D(t)$ the set of regular solutions of problem $(\mathscr{P})$ with initial time t.

It is easy to obtain the following

PROPOSITION 4. *For every* $t \in [0, T]$

$$\delta_{D(t)} \geq \Gamma(N^+, \mathscr{B}^+, (L^1)^+, (L^\infty)^-) \lim_h \delta_{\mathscr{A}_h(t)}$$

that is, if $(b, U, q) \in D(t)$ then, for every $b_h \to b$ in $\mathscr{B}$, for every $U_h \to U$ in L^1 there exists $q_h \to q$ in L^∞ such that $(b_h, U_h, q_h) \in \mathscr{A}_h(t)$ for all h large enough.

The crucial point in the proof of property II) consists in proving the following

THEOREM 5. *For every* $t \in [0, T]$ $D(t)$ *is dense in* $\mathscr{A}(t)$ *with respect to the product topology of* $\mathscr{B} \times L^1 \times L^\infty$.

The thesis of Theorem 1 follows now from the previous results and from general Γ-convergence results (see [4], [8], [9]).

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