Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Giulio Maier

Inverse problem in engineering plasticity: a quadratic programming approach

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **70** (1981), n.4, p. 203–209.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1981_8_70_4_203_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1981.

Meccanica dei solidi e delle strutture. — *Inverse problem in engineering plasticity: a quadratic programming approach* (*). Nota di GIULIO MAIER (**), presentata (***) dal Corrisp. L. FINZI.

RIASSUNTO. — Si considera un modello discreto (per elementi finiti) di un solido o un sistema strutturale perfettamente elastoplastico, con condizioni di snervamento « linearizzate a tratti », nell'ipotesi di olonomia assunta per processi di caricamento proporzionali. Supponendo noti su base sperimentale certi spostamenti sotto assegnate azioni esterne, si formula il problema di identificare i limiti di snervamento, ossia le resistenze locali. Si dimostra che questo problema inverso di meccanica strutturale non lineare è trasformabile in un equivalente programma quadratico non convesso, suscettibile di risoluzione relativamente agevole con varie tecniche numeriche.

1. INTRODUCTION

An inverse problem in engineering mechanics can be formulated as follows, in sufficiently general terms for the present purposes: in a mathematical model of a mechanical system some parameters which characterize physical and/or geometrical properties are regarded as unknowns, besides the usual variables (such as displacements and stresses) which define the system response to given external actions (loads); experimental data on the response of the system to the given loads compensate for the lacking information on the system parameters; these are sought by minimizing a suitable measure of the discrepancy between the experimental data and the corresponding quantities predicted by the model for the same loads.

The practical motivation arises from the fact that in some structural and geotechnical engineering situations certain properties are not susceptible to direct in situ or laboratory measurements, see e.g. [7].

From the engineering mechanics standpoint the conceptual interest of this class of system identification problems rests on its distinct and relatively novel features with respect to the traditional classes of analysis problems (where all unknowns concern the response to external actions) and design problems (where

(*) This work is part of a research project supported by C.N.R., Gruppo P.A.dI.S.
(**) Istituto di Scienza e Tecnica delle Costruzioni del Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano (Tel 02/29.56.64).
(***) Nella seduta del 24 aprile 1981.

unknowns include also system characteristics, but conditions are imposed on the system response and a merit or cost function has to be optimized).

The inverse problem studied in this paper concerns the local resistances (or yield limits) in a discrete (finite element) model of an elastic-perfectly plastic, holonomic system with piecewiselinear yield loci; it presumes experimental information on the displacement response to quasi-static loading. The expression engineering plasticity (as distinct from continuum plasticity) refers to the discrete, computer-oriented nature of the model. The hypothesis of holonomic, path-independent constitutive laws (in the spirit of the " deformation theory " of plasticity) is justified in practice by the possibility of choosing for the experiments proportional loadings (which make local unstressing and irreversibility manifestations unlikely or negligible); this assumptions leads to a computationally convenient model in finite, instead of incremental, terms [2, 9]. The present approach is deterministic: errors (or " noises ") possibly affecting measurements and/or model might be considered separately by estimation and filtering techniques [3].

Related previous works have dealt with the identification of elastic moduli in linear structural models [6] and of plastic properties by a purely numerical, direct-search approach, implying a large number of solutions to corresponding analysis problems [5]. The present structural identification problem, associated to the above specified class of nonlinear models, making use of their analytical peculiarities, is shown here to be ameneable to a single quadratic programming problem and, hence, to require presumably a computational effort comparable to that of a single corresponding structural analysis problem. Computational aspects, alternative approaches, further developments and numerical tests are presented elsewhere [10].

2. Formulation of the model

The class of discrete structural models considered will be such that the compatibility and equilibrium equations for the element aggregate can be expressed in the form:

$$\underline{q} = \underline{C}\,\underline{u} \tag{1}$$

$$\tilde{\underline{C}}\,\underline{Q} = \underline{F} \tag{2}$$

respectively, where: $\underline{u}$ and $\underline{F}$ denote vectors of the free nodal displacements (degrees-of-freedom) and corresponding nodal loads; $\underline{q}$ and $\underline{Q}$ are vectors of element generalized strains and stresses defined in the " natural " sense (i.e. unaffected by rigid-body motions, and selfequilibrated, respectively) [1]; matrix $\underline{C}$ depends on the undeformed geometry only; the tilde means transpose. Eqs. (1) (2) entail the usual " small deformations " assumption. The individual behaviours of all constituents of the (disassembled) model are described as follows:

(3 a) $$\underline{Q} = \underline{S}\,(\underline{q} - \underline{N}\,\underline{\lambda})$$

(3 b) $$\underline{\phi} = \underline{\tilde{N}}\,\underline{Q} - \underline{r} \leq \underline{0}$$

(3 c) $$\underline{\lambda} \geq \underline{0} \quad , \quad \underline{\tilde{\phi}}\,\underline{\lambda} = 0$$

where: $\underline{S}$ denotes the block-diagonal matrix of the (symmetric, positive definite) elastic stiffness matrices of all elements; $\underline{\phi}$, $\underline{\lambda}$ and $\underline{r}$ are n-vectors of yield functions, " plastic multipliers " and yield limits, respectively, for all yield planes in the generalized stress spaces of all constituents; $\underline{N}$ indicates a block-diagonal matrix whose diagonal blocks are formed by columns representing the outward normal unit vectors to the yield planes of the relevant finite element or structural constituents; $\underline{0}$ is a vector of zeros; vectors inequalities apply componentwise. The relation set (3) defines piecewise-linear, holonomic, elastic—perfectly plastic constitutive laws for all elements [2, 9].

By substituting vectors $\underline{q}$, $\underline{Q}$, $\underline{u}$, the above set of governing relations can be given the form of the following linear complementarity problem [8]:

(4 a) $$\underline{\phi} = \underline{\tilde{N}}\,\underline{Q}^e + \underline{\tilde{N}}\,\underline{Z}\,\underline{N}\,\underline{\lambda} - \underline{r}$$

(4 b) $$\underline{\phi} \leq \underline{0} \quad \lambda \geq \underline{0} \quad , \quad \underline{\tilde{\phi}}\,\underline{\lambda} = 0$$

where: $\underline{Q}^e$ represents a known vector of the linear elastic stress response to the given loads $\underline{F}$; $\underline{Z}$ is a (symmetric, negative semidefinite) matrix of influence coefficients for stresses due to imposed strains in the elastic range.

3. THE IDENTIFICATION PROBLEM

Let the yield limits depend linearly on p unknown parameters collected in vector $\underline{P}$, usually constrained by bounds to a domain of physical significance:

(5) $$\underline{r} = \underline{R}\,\underline{P} \quad , \quad \underline{P}^L \leq \underline{P} \leq \underline{P}^U .$$

Some, say $m \geq p$, nodal displacements in the real system under the given loads modeled by $\underline{F}$ are assumed to be known through measurements and form vector $\underline{u}^m$. The vector $\underline{u}^c$ of the same quantities calculated by means of the mathematical model of Sec. 2 can be expressed as:

(6) $$\underline{u}^c = \underline{u}^e + \underline{G}\,\underline{\lambda}$$

where the known vector $\underline{u}^e$ contains the corresponding displacements which would be provoked by $\underline{F}$ in a hypothetical linear elastic regime; matrix $\underline{G}$ transforms into " plastic displacements " vector $\underline{\lambda}$ which corresponds to $\underline{r}$ through (4). $\underline{G}$, $\underline{Z}$, $\underline{Q}^e$, $\underline{u}^e$ are obtained by easy manipulations of (1)–(3), as functions of $\underline{S}$ and $\underline{C}$ ($\underline{Q}^e$ and $\underline{u}^e$ also of $\underline{F}$).

The discrepancy between measured and calculated displacements, i.e. the "loss function", can be defined as:

$$\omega \equiv (\tilde{\underline{u}}^c - \tilde{\underline{u}}^m)\, \underline{D}\, (\underline{u}^c - \underline{u}^m) \tag{7}$$

where $\underline{D}$ is a diagonal matrix of (positive) weighting coefficients taking into account possible differences in confidence level among the measured quantities.

Substituting (6) in (7) provides a quadratic form in $\underline{\lambda}$ associated to a symmetric, positive semidefinite matrix $\underline{\underline{M}}$. The constrained minimization of such an expression of the loss function with respect to $\underline{P}$, $\underline{\lambda}$, $\underline{\phi}$ provides a formulation of the inverse problem in point as a problem in nonconvex nonquadratic programming, namely:

$$\min \{\omega(\underline{\lambda}) \equiv \tilde{\underline{\lambda}}\, \underline{\underline{M}}\, \underline{\lambda} + \tilde{\underline{b}}\, \underline{\lambda} + c\} \tag{8 a}$$

(8 b) subject to: (4) and (5)

where $\underline{\underline{M}}$ and $\underline{b}$ are a vector and a matrix, respectively, of data and c a given constant. The difficulty of solving numerically this problem arises primarily from the presence of the (nonconvex) complementarity constraint (4 b) and motivates the transformation established in the next Section.

4. Reduction to quadratic programming

It will be proved below that problem (8) is equivalent to the following nonconvex quadratic program:

$$\min \{\psi(\underline{\lambda}, \underline{\phi}) \equiv \omega(\underline{\lambda}) - \rho\, \tilde{\underline{\phi}}\, \underline{\lambda}\} \tag{9 a}$$

subject to:

$$\underline{\phi} = \tilde{\underline{N}}\, \underline{Q}^e + \tilde{\underline{N}}\, Z\, \underline{N}\, \underline{\lambda} - \underline{R}\, \underline{P} \leq \underline{0} \tag{9 b}$$

$$\underline{\lambda} \geq \underline{0} \quad , \quad \underline{P}^L \leq \underline{P} \leq \underline{P}^U \tag{9 c}$$

where ρ is a positive real to be chosen not lesser than a suitable threshold valued ρ_0.

Recourse is made to a theorem established in [4]. This is re-stated here for convenience, denoting by $\|\cdot\|$ the Euclidean norm, by f and χ real-valued functions, by Y a closed set, by X and Z compact sets of the Euclidean space such that $Z \subset X$.

THEOREM [4]. Let the following conditions be fulfilled:

(*a*) function f is bounded on X and there exists an open set $\Omega \supset Z$ and real numbers α, $\beta > 0$, such that for any $\underline{x}, \underline{y} \in \Omega$, the following Hölder inequality holds:

$$|f(\underline{x}) - f(\underline{y})| \leq \alpha \|\underline{x} - \underline{y}\|^\beta \tag{10}$$

(*b*) it is possible to find a function χ such that: (*i*) χ is continuous on X; (*ii*) $\chi(\underline{x}) = 0$ for $\underline{x} \in Z$, $\chi(\underline{x}) > 0$ for $\underline{x} \in X - Z$; (*iii*) for any $\underline{z} \in Z$ there exists a real $\bar{\varepsilon} > 0$ and a neighbourhood say $S(\underline{z})$ of $\underline{z}$, such that, for any $x \in S(\underline{z}) \cap (X - Z)$:

$$\chi(\underline{x}) \geq \bar{\varepsilon} \, \| \underline{x} - \underline{z} \|^{\beta} . \tag{11}$$

The a real ρ_0 exists such that for any $\rho \geq \rho_0$ the problem

$$\min f(\underline{x}) \,, \quad \text{subject to:} \quad \underline{x} \in Z \cap Y \tag{12}$$

is equivalent to the problem

$$\min \{ f(\underline{x}) + \rho \, \chi(\underline{x}) \} \,, \quad \text{subject to:} \quad x \in X \cap Y . \tag{13}$$

In order to derive from the above theorem the equivalence between problems (8) and (9), let us first define the following sets of vectors $(\underline{\lambda}, \underline{\phi}, \underline{P})$:

$$U \equiv \{ (\underline{\lambda}, \underline{\phi}, \underline{P}) : \underline{0} \leq \underline{\lambda} \leq \underline{\Lambda} \;\; ; \;\; -\underline{\Phi} \leq \underline{\phi} \leq \underline{0} \;\; ; \;\; \underline{P}^{L} \leq \underline{P} \leq \underline{P}^{U} \} \tag{14}$$

$$V \equiv \{ (\underline{\lambda}, \underline{\phi}, \underline{P}) : \underline{\phi} = \underline{\tilde{N}} \, \underline{Q}^{e} + \underline{\tilde{N}} \, Z \, \underline{N} \, \underline{\lambda} - \underline{R} \, \underline{P} \} \tag{15}$$

$$W \equiv \{ (\underline{\lambda}, \underline{\phi}, \underline{P}) \subset X : \underline{\tilde{\phi}} \, \underline{\lambda} = 0 \} \tag{16}$$

where $\underline{\Lambda}$ and $\underline{\Phi}$ are vectors of upper bounds such that they are not reached by the corresponding variables in the solution, a condition easily complied with by engineering judgement in all practical situations, as it is implicitly assumed that the loads $\underline{F}$ do not exceed the carrying capacity, i.e. that a (bounded) solution to (4) exists for the actual values of the parameters. Thus problems (8) and (9) can be reformulated in the following forms, respectively:

$$\min \omega(\underline{\lambda}) \,, \quad \text{subject to:} \;\; (\underline{\lambda}, \underline{\phi}, \underline{P}) \in V \cap W \tag{17}$$

$$\min \{ \omega(\underline{\lambda}) - \rho \, \underline{\tilde{\phi}} \, \underline{\lambda} \} \,, \quad \text{subject to:} \;\; (\underline{\lambda}, \underline{\phi}, P) \in U \cap V . \tag{18}$$

Clearly, the sets U and W are compact, $W \subset U$ and V is closed. The compacteness of U and the continuity of $\omega(\underline{\lambda})$ ensures the boundedness of this function on U. Let Ω be an open sphere containing W, and let Ω_λ be the intersection of Ω with the subspace of vectors $\underline{\lambda}$. Taking into account (8 a) and the Cauchy inequality, one realizes that, for any pair of vectors $\underline{\lambda}', \underline{\lambda}'' \in \Omega_\lambda$:

$$\begin{aligned} \omega(\underline{\lambda}') - \omega(\underline{\lambda}'') &= (\underline{\tilde{\lambda}}' - \underline{\tilde{\lambda}}'') \, [\underline{M} (\underline{\lambda}' - \underline{\lambda}'') + \underline{b}) \leq \\ &\leq \| \lambda' - \lambda'' \| \cdot \| \underline{M} (\lambda' - \underline{\lambda}'') + \underline{b} \| \leq \| \lambda' - \lambda'' \| \, \gamma \end{aligned} \tag{19}$$

having set:

$$\gamma \equiv \sup_{\lambda \in \Omega_\lambda} \| \underline{M} \, \underline{\lambda} + b \| < + \infty . \tag{20}$$

Therefore, the Hölder inequality (10) is fulfilled by function $\omega(\underline{\lambda})$ for $\alpha = \gamma$ and $\beta = 1$, and, hence, condition (*a*) turns out to be satisfied when sets X and W are identified with the above defined U and W, respectively.

As for condition (*b*), its parts (*i*) and (*ii*) are immediately seen to hold for the function $\chi = -\tilde{\underline{\phi}}\,\underline{\lambda}$ if $X = U$ and $Z = W$. We prove below that also part (*iii*) holds for $\beta = 1$, when $X = U$ and $Z = W$. Note that the interior of set W is empty. Setting $\underline{x} \equiv (\underline{\lambda}, -\underline{\phi}) \in U$ and $\underline{z} \equiv (\underline{\lambda}', -\underline{\phi}') \in W$, let us define:

$$S(\underline{z}) \equiv \{\underline{x} : \delta \equiv \|\underline{x} - \underline{z}\| < \bar{\delta}\} \tag{21}$$

$$(\underline{\xi}, \underline{\eta}) \equiv 1/\delta\,(\underline{\lambda} - \underline{\lambda}', -\underline{\phi} + \underline{\phi}') = 1/\delta\,(\underline{x} - \underline{z}) \tag{22}$$

where $\bar{\delta}$ is a positive real.

Since $\tilde{\underline{\phi}}'\,\underline{\lambda}' = 0$, one can write:

$$-\tilde{\underline{\phi}}\,\underline{\lambda} = \delta^2\,\tilde{\underline{\xi}}\,\underline{\eta} + \delta\,(-\tilde{\underline{\phi}}'\,\underline{\xi} + \tilde{\underline{\lambda}}'\,\underline{\eta}) = \delta\,[\delta\,\tilde{\underline{\xi}}\,\underline{\eta} + \Sigma\xi_i(-\phi_i') + \Sigma\eta_i\lambda_i') \tag{23}$$

where the former summation concerns only those components i's such that $\lambda_i' = 0$, $\phi_i' < 0$, the latter is restricted to those i's such that $\lambda_i' > 0$. Let $\Rightarrow$ mean implication; we notice that:

$$\xi_i < 0 \Rightarrow \lambda_i' > 0 \Rightarrow \phi_i' = 0 \Rightarrow \eta_i \geq 0 \tag{24 a}$$

$$\eta_i < 0 \Rightarrow \phi_i' < 0 \Rightarrow \lambda_i' = 0 \Rightarrow \xi_i \geq 0\,. \tag{24 b}$$

Therefore, the summation in (23) contain nonnegative ξ_i, η_i and, hence, if $\underline{z} \neq \underline{0}$, their sum is positive (as $\underline{\xi} = \underline{\eta} = 0$ cannot occur) and has a positive minimum, say k, over the intersection of the sphere $\|(\underline{\xi}, \underline{\eta})\| = 1$ with the nonnegative orthant $\underline{\xi} \geq \underline{0}$, $\eta \geq \underline{0}$. In fact, this sum is a (homogeneous) linear form whose gradient has positive components. The minimum of $\tilde{\underline{\xi}}\,\underline{\eta}$ over $\|(\underline{\xi}, \underline{\eta})\| = 1$ is -1. As a consequence, it follows from (23) that:

$$-\tilde{\underline{\phi}}\,\underline{\lambda} \geq \delta\,(-\delta + k) > \delta\,(-\bar{\delta} + k) = \bar{\varepsilon}\,\|(\lambda, -\phi) - (\lambda', -\phi')\|\,. \tag{25}$$

Since k does not depend on $\bar{\delta}$, $\bar{\delta}$ can be chosen sufficiently small to make $\bar{\varepsilon} \equiv k - \bar{\delta}$ positive; then (25) becomes (11) with $\beta = 1$. This completes the proof that also condition (b) is fulfilled. Thereafter, since the vector set V (15) is closed and can be identified with Y referred to in the above theorem, this theorem ensures the equivalence between problems (17) and (18), i.e. between problems (8) and (9). An expression for ρ_0 established in [4] can be easily applied to the present context [10].

5. CONCLUSIONS

The transformation of the inverse problem formulated in Sect. 3 to a quadratic program, by virtue of the equivalence demonstrated in Sect. 4, appears to imply significant computational gains, although the quadratic program is not convex. As it will be shown in [10], these advantages rest primarily on the circumstances that at least a Karush-Kuhn-Tucker point can be efficiently obtained by a familiar algorithm for convex quadratic programming. This point can be checked for optimality by means of various criteria; it turns out to correspond to the global minimum in most practical cases, according to the numerical experience achieved so far; otherwise it can be used to initialize a second phase leading to the global minimum.

Acknowledgements. The author gratefully acknowledges the friendly advice of F. Giannessi.

REFERENCES

[1] J. H. ARGYRIS, *et alii* (1979) – *Finite Element Method: the Natural Approach,* « Comp. Meth. Appl. Mech. and Engrg. » Vol. 17/18, 1–106.

[2] M. Z. COHN and G. MAIER Eds., (1979) – « Engineering Plasticity by Mathematical Programming », Pergamon Press. New York.

[3] P. EYKHOFF, (1974) – *System identification,* J. Wiley, Chichester.

[4] F. GIANNESSI and F. NICOLUCCI (1976) – *Connections between Nonlinear and Integer Programming Problems,* « Ist. Naz. Alta Matematica », Symposia Mathematica, Vol. 19, Academic Press. 161–176.

[5] G. GIODA and G. MAIER (1980) *Direct Search Solution of an Inverse Problem in Elastoplasticity: Identification of Cohesion, Friction Angle and in Situ Stress by Pressure Tunnel Tests,* « Int. J. Num. Methods in Engrg. », Vol. 15, No. 12, 1823–1848.

[6] K. T. KAVANAGH (1973) – *Experiment vs. Analysis: Computational Techniques for the Description of Static Material Response,* « Int. J. Num. Methods in Engrg. », Vol. 5, 503–515.

[7] S. C. LIN and J. T. P. YAO (1978) – *Structural Identification Concept,* « J. Struct. Div. Proc. ASCE », Vol. 104, No. 12, 1845–1858.

[8] G. MAIER (1968) – *A Quadratic Programming Approach for Certain Classes of Nonlinear Structural Problems,* « Meccanica », Vol. 3, No. 2, 121–130.

[9] G. MAIER (1973) – *Mathematical Programming Methods in Structural Analysis,* H. Tottenham and C. Brebbia, Eds., « Variational Methods in Engineering », Southampton Univ. Press.

[10] G. MAIER, F. GIANNESSI and A. NAPPI (1982) – *Indirect Identification of Yield Limits by Mathematical Programming,* « Engineering Structures », to appear.