Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Solutions in Gevrey spaces of partial differential equations with constant coefficients

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **70** (1981), n.3, p. 137–142.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1981_8_70_3_137_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1981.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 14 marzo 1981

Presiede il Presidente della Classe GIUSEPPE MONTALENTI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *Solutions in Gevrey spaces of partial differential equations with constant coefficients.* Nota di LAMBERTO CATTABRIGA (*), presentata (**) dal Socio G. CIMMINO.

RIASSUNTO. — Si dà una condizione sufficiente per la esistenza di una soluzione in uno spazio di Gevrey $\Gamma^d(\mathrm{R}^n)$, d razionale ≥ 1, $n \geq 2$, di una equazione lineare a derivate parziali a coefficienti costanti $\mathrm{P}(\mathrm{D})\, u = f$, quando $f \in \Gamma^d(\mathrm{R}^n)$. La dimostrazione completa dei risultati ottenuti è contenuta in una nota dell'autore in corso di pubblicazione su " Astérisque ".

1. In this paper we give a sufficient condition for the existence of a solution u in a Gevrey space $\Gamma^d(\mathrm{R}^n)$, d a rational number ≥ 1, $n \geq 2$, to a linear partial differential equation with constant coefficients $\mathrm{P}(\mathrm{D})\, u = f$, when $f \in \Gamma^d(\mathrm{R}^n)$. The result which we state here and the method for its proof may be viewed as an extension of results and methods contained in previous papers by E. De Giorgi and the author [7], [8], [5], [6], [3], [4] concerning the case $d = 1$. For this case see also [1], [12], [13].

By $\Gamma^d(\mathrm{R}^n)$, $d > 0$ we denote the set of all C^∞ complex valued functions f in R^n such that for every compact set $\mathrm{K} \subset \mathrm{R}^n$ there exists a constant $c(\mathrm{K})$, which depends on f, such that

$$\sup_{x \in \mathrm{K}} |\mathrm{D}^\alpha f(x)| \leq c(\mathrm{K})^{|\alpha|+1}\, \Gamma(d\,|\alpha| + 1), \qquad \alpha \in \mathrm{Z}^n_+$$

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(**) Nella seduta del 14 marzo 1981.

where Γ is the Euler gamma function. Here D stands for $(D_1, \cdots, D_n)$, with $D_j = -i\,\partial/\partial x_j$, $j = 1, \cdots, n$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Here is our main result.

THEOREM 1.1 (1). *Let* P (D) *be a linear differential operator with constant coefficients and let* $d \geq 1$ *be a given rational number. Suppose that there exists a finite number of vectors* $N^j \in R^n \setminus \{0\}$, $j = 1, \cdots, l$, *such that*

a) *for every* $j = 1, \cdots, l$

$\xi \in R^n$, $t \in R$, $|\xi - (\xi, N^j) N^j| \geq k$, $P(\xi + it N^j) = 0$ *imply either*

$$t \geq -c_1 |\xi - (\xi, N^j) N^j|^{1/\rho} \quad \text{or} \quad t \leq -c_2 |\xi|^{1/d}$$

for some constants $k > 1$, $\rho > d$, $c_1 > 0$, $c_2 > 0$;

b) *there exist positive constants* γ_j, $j = i, \cdots, l$ *such that*

$$R^n \setminus \{0\} = \bigcup_{j=1}^{l} \Delta_j$$

where $\Delta_j = \{y \in R^n ; |y| < \gamma_j (y, N^j)\}$, $j = 1, \cdots, l$.

Then $P(D)\, \Gamma^d(R^n) = \Gamma^d(R^n)$.

Remarks 1.2.

i) Condition a) in Theorem 1.1 is in particular satisfied if the polynomial $Q(\eta', \eta_n)$, $\eta' = (\eta_1, \cdots, \eta_{n-1})$, obtained from P by a change of orthogonal coordinates which sends η_n over N^j, is $\binom{n}{p}$-*hypoelliptic* of *exponent* $1/d$ in the sense of E. A. Gorin [10], for every $p = 1, \cdots, n$.

ii) Conditions a) and b) are obviously satisfied by every *d'-hypoelliptic polynomial* P, with $1 \leq d' \leq d$ and by every polynomial P, ρ-hyperbolic in the sense of E. Larsson [14] with respect to a vector $N \in R^n \setminus \{0\}$ (and hence with respect to every vector in some open cone containing N), when $d < \rho \leq \infty$.

iii) All polynomials considered by J. Fehrman [9], which are called *hybrids between hyperbolic and elliptic polynomials* also satisfy conditions a) and b) of Theorem 1.1, for any $d \geq 1$.

iv) The conclusion of Theorem 1.1 is obviously true for all polynomials which may be written as a product of polynomials of the types considered in *ii*) and *iii*). In particular this is the case for *every homogeneous polynomial in two variables and any* $d \geq 1$.

2. The proof of Theorem 1.1 is based on a representation formula for any $f \in \Gamma^d(R^n)$ and on the construction of a particular solution of the equation $P(D)\, v = G_1$, where G_1 is a kernel depending on a parameter, connected with the representation formula of any $f \in \Gamma^d(R^n)$.

(1) When $d = 1$ see [4], Theorem 4.6.

Let $d = r/s$, $r \geq s$ positive natural numbers, and let $\bar{n} \geq 1$ be a natural number such that $\frac{n}{2s} + \frac{\bar{n}}{2r} > 1$. Put

$$Q(\xi, \tau) = |\xi|^{2s} + |\tau|^{2r}, \qquad \xi \in R^n, \ \tau \in R^{\bar{n}}$$

and let E be the distribution on $R^{n+\bar{n}}$ defined by

$$(2.1) \qquad \langle E, \phi \rangle = (2\pi)^{-(n+\bar{n})} \int_0^{+\infty} d\nu \int_{R^{n+\bar{n}}} e^{-\nu Q(\xi,\tau)} \hat{\phi}(-\xi, -\tau) \, d\xi \, d\tau, \qquad \phi \in C_0^\infty(R^{n+\bar{n}}),$$

where $\hat{\phi}(\xi, \tau) = (\mathscr{F}\phi)(\xi, \tau) = \int_{R^{n+\bar{n}}} e^{-i(\langle x, \xi \rangle + \langle t, \tau \rangle)} \phi(x, t) \, dx \, dt$

is the Fourier transform of ϕ. It can be easily proved that $E \in C^\infty(R^{n+\bar{n}} \setminus \{(0, 0)\})$ and that

$$(2.2) \qquad E(x, t) = \int_0^{+\infty} E(x, t; \nu) \, d\nu, \qquad (x, t) \in R^{n+\bar{n}} \setminus \{(0, 0)\},$$

where $E(x, t; \nu) = \mathscr{F}^{-1}_{(\xi,\tau)}(\exp(-\nu Q(\xi, \tau)))(x, t)$.

It turns out that E is a fundamental solution to the semi-elliptic operator $Q(D_x, D_t)$ and that for every $\nu > 0$, $E(x, t; \nu)$ is a radial function with respect to $x \in R^n$ and with respect to $t \in R^{\bar{n}}$.

The following representation theorem holds.

THEOREM 2.1 (2). *Let $f \in \Gamma^d(R^n)$, $d = r/s$, $r \geq s$ positive natural numbers, and let ϕ be a given positive non increasing function on* R. *Then there exists a function $g \in C^\infty(R^n \times R^+)$ and a positive, non increasing C^∞ function ψ on* R *such that*

$$f(x) = \int_{R^n} dy \int_0^{+\infty} G(x - y, \sigma) g(y, \sigma) \, d\sigma, \qquad x \in R^n,$$

$$\operatorname{supp} g \subset \{(y, \sigma) \in R^n \times R^+ ; \psi(|y|^2) \leq \sigma \leq 2\psi(|y|^2)\},$$

$$\int_0^{+\infty} |g(y, \sigma)| \, d\sigma \leq \phi(|y|^2)$$

(2) When $d = 1$ see [7].

where $G(x, \sigma) = E(x, t)$, *for* $|t| = \sigma > 0$, $E(x, t)$ *given by* (2.1), (2.2) *when* $\bar{n} > 2r$.

Remark 2.2. Let $\nu_0 > 0$ and for $x \in R^n$, $\sigma > 0$ put

$$G_1(x, \sigma) = \int_0^{\nu_0} E(x, t; \nu)\Big|_{|t|=\sigma} d\nu \quad , \quad G_2(x, \sigma) = \int_{\nu_0}^{+\infty} E(x, t; \nu)\Big|_{|t|=\sigma} d\nu .$$

It is easily seen that there exists a positive constant c such that

$$|D_x^\alpha G_2(x, \sigma)| \leq c^{|\alpha|+1} \Gamma(|\alpha|/_{2s} + 1), \qquad x \in R^n, \ \sigma > 0, \ \alpha \in Z_+^n .$$

This implies that if the function ϕ in Theorem 2.1 is chosen so that $\int_{R^n} \phi(|y|^2) \, dy < \infty$, the function

$$f_2(x) = \int_{R^n} dy \int_0^{+\infty} G_2(x - y, \sigma) \, g(y, \sigma) \, d\sigma, \qquad x \in R^n$$

belongs to the space $\Gamma^{1/2s}(R^n)$ and thus to every space $\gamma^\delta(R^n)$, with $\delta > 1/2s$. Here $\gamma^\delta(R^n)$ is the space of all C^∞ function f on R^n such that for every compact set $K \subset R^n$ and every $\varepsilon > 0$

$$\sup_{\alpha \in Z_+^n} \varepsilon^{-|\alpha|} \Gamma(\delta|\alpha| + 1)^{-1} \sup_{x \in K} |D_x^\alpha f(x)| < \infty .$$

The following theorem will enable us to find a solution in $\Gamma^d(R^n)$ of the equation $P(D) u = f_1$, where $f_1 = f - f_2$.

THEOREM 2.3. *Let* $P(D) = \sum_{j=0}^m a_j(D') D_n^j$, $D' = (D_1, \cdots, D_{n-1})$, *be a linear differential operator with constant coefficients and let* $d = r/s \geq 1$, $\bar{n} > 2r$, $0 < \nu_0 < 1$.

Assume that there exist constants $k > 1$, $\rho > d$, $c_1 > 0$, $c_2 > 0$ *such that*

$$(2.3) \quad \begin{array}{l} (\xi', \lambda) \in R^{n-1} \times C \quad , \quad |\xi'| > k, \quad P(\xi', \lambda) = 0 \quad \textit{imply} \\ \textit{either} \quad \operatorname{Im}\lambda \geq -c_1 |\xi'|^{1/\rho} \quad \textit{or} \quad \operatorname{Im}\lambda \leq -c_2(|\xi'| + |\operatorname{Re}\lambda|)^{1/d}. \end{array}$$

Then for every $\sigma > 0$ *there exists a solution* $v(\cdot, \sigma) \in C^\infty(R^n)$ *to the equation* $P(D) v = G_1(x, \sigma)$ *such that for every* $\alpha \in Z_+^n$

$$|D_x^\alpha v(x, \sigma)| \leq c^{|\alpha|+1} e^{c'|x|} \sigma^{-d|\alpha|} \Gamma(d|\alpha| + 1) \int_0^{\nu_0} \nu^{-\frac{m+n+q}{2s} - \frac{\bar{n}}{2r}} \cdot$$

$$\cdot \exp\left[-c'' \nu^{-1/(2r-1)} \left[\sigma^{2r/(2r-1)} - \tilde{c}''(1 + |x_n|)^{\frac{2s\rho}{2s\rho - 1}} \nu^{\frac{2s\rho - 2r}{(2s\rho - 1)(2r-1)}}\right]\right] d\nu$$

for any $x \in \mathrm{R}^n$, *and* $|D_x^\alpha v(x, \sigma)| \leq c^{|\alpha|+1} e^{c'|x|} \Gamma(d|\alpha|+1)$ *for any* $x \in \mathrm{R}^n$ *with* $x_n < \delta$, $\delta < -\frac{\bar{n}}{2r} + 1$, *where* $c, c', c'', \tilde{c}''$ *are positive constants independent of* x, σ, α *and* q *is a non negative number such that* $\sum_{j=0}^{m} |\alpha_j(\xi')| \geq c_3 |\xi'|^{-q}$, $|\xi'| > k+1$, *for some positive constant* c_3 (3).

The proofs of Theorems 2.1 and 2.3 will appear on "Astérisque".

3. The following theorem is now easy to prove.

THEOREM 3.1. *Let* P(D) *satisfy the assumptions of Theorem 2.3 and let*

$$h(x) = \int_{\mathrm{R}^{n+1}} G_1(x-y, \sigma)\, g(y, \sigma)\, dy\, d\sigma, \qquad x \in \mathrm{R}^n$$

where $g \in C^\infty(\mathrm{R}^{n+1})$ *and*

i) supp $g \subset \{(y, \sigma) \in \mathrm{R}^{n+1};\ y_n \geq c_0, \sigma \geq \chi(y_n)\}$, c_0 *a constant,* χ *a positive continuous function on* R;

ii) $\int_{\mathrm{R}^{n+1}} e^{c'|y|} |g(y, \sigma)|\, dy\, d\sigma < \infty$, *where* c' *is the same constant as in the estimates for the function* v *in Theorem 2.3.*

Then $h \in \Gamma^d(\mathrm{R}^n)$ *and the function* $u(x) = \int_{\mathrm{R}^{n+1}} v(x-y, \sigma)\, g(y, \sigma)\, dy\, d\sigma$, $x \in \mathrm{R}^n$, v *as in Theorem 2.3, is such that* $u \in \Gamma^d(\mathrm{R}^n)$ *and* $\mathrm{P(D)}\, u = h$.

Proof of Theorem 1.1. Let $\{\chi_j\}$, $j = 0, \cdots, l$ be a C^∞ partition of unity subordinate to the covering $\{\Delta_0, \Delta_1, \cdots, \Delta_l\}$ of R^n, Δ_0 an open ball centered at the origin of R^n, Δ_j, $j = 1, \cdots, l$, the open cones of condition *b*), and let

$$f_1(x) = \sum_{j=0}^{l} \int_{\mathrm{R}^n} dy \int_0^{+\infty} G_1(x-y, \sigma)\, \chi_j(y)\, g(y, \sigma)\, d\sigma = \sum_{j=0}^{l} h_j(x),$$

where g is the same function as in the representation formula for f of Theorem 2.1, with ϕ chosen so that $\int_{\mathrm{R}^n} e^{c'|y|} \phi(|y|^2)\, dy < \infty$, c' the same constant as in Theorem 2.3. Application of Theorem 3.1, after a rotation of coordinates and a translation when $j = 0$, leads to a solution in $\Gamma^d(\mathrm{R}^n)$ for each equation $\mathrm{P(D)}\, u = h_j$, $j = 0, \cdots, l$, and hence to a solution $u_1 \in \Gamma^d(\mathrm{R}^n)$

(3) The existence of a number q with this property follows from assumption (2.3) and from a well-known lemma by L. Hörmander [11].

of the equation $P(D)u = f_1$. Since the function $f_2 = f - f_1$ defined in Remark 2.2, belongs to $\gamma^d(R^n)$, by well known results (4) there exists a function $u_2 \in \gamma^d(R^n)$ such that $P(D)u_2 = f_2$. Thus the function $u = u_1 + u_2$ is a solution in $\Gamma^d(R^n)$ of the equation $P(D)u = f$.

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(4) See B. Malgrange [15] when $d = 1$, F. Treves [16] and G. Björck [2] when $d > 1$.