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Eugenio Sinestrari

Abstract semilinear equations in Banach spaces

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Analisi matematica. — *Abstract semilinear equations in Banach spaces* (*). Nota di EUGENIO SINESTRARI, presentata (**) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si studiano le proprietà delle soluzioni dell'equazione semilineare astratta $u'(t) = \Lambda u(t) + \varphi(t, u(t))$ quando Λ è il generatore infinitesimale di un semigruppo analitico in uno spazio di Banach. Vengono provati nuovi teoremi di regolarità anche nel caso in cui φ non è continuo in tutto lo spazio.

1. INTRODUCTION

We shall be concerned with the following abstract semilinear differential equation

$$(1) \qquad \begin{cases} u'(t) = \Lambda u(t) + \varphi(t, u(t)) \\ u(0) = x \end{cases}$$

where $\Lambda : D_\Lambda \subset E \to E$ is the infinitesimal generator of a holomorphic semigroup $e^{\Lambda t}$ in the Banach space E and $\varphi(t, x)$ is a continuous function from $[0, T] \times E$ to E. The case in which φ is not continuous is considered in the last section.

DEFINITION 1. A function $u \in C^1(0, T; E) \cap C(0, T; D_\Lambda)$ (D_Λ is endowed with the graph norm) is a *strict solution* of (1) if (1) holds for each $t \in [0, T]$.

Let us set $X = C(0, T; E)$, $D_A = C(0, T; D_\Lambda)$ and let $A : D_A \subset X \to X$, $(Au)(t) = \Lambda u(t)$ be the multiplication operator of Λ in X ([5]). For each $u \in X$ set $(Fu)(t) = \varphi(t, u(t))$, $0 \leq t \leq T$.

DEFINITION 2. A function $u \in C(0, T; E)$ is a *strong solution* of (1) if there exists $u_n \in C^1(0, T; E) \cap C(0, T; D_\Lambda)$ for each $n \in \mathbf{N}$ such that 1) $\lim_{n \to \infty} u_n = u$ in X, 2) $\lim_{n \to \infty} [u'_n - Au_n - F(u_n)] = 0$ in X, 3) $\lim_{n \to \infty} u_n(0) = x$ in E.

Remark 1. As $F : X \to X$ is continuous, condition 2) can be replaced by: 2') $\lim_{n \to \infty} [u'_n - Au_n] = F(u)$ in X.

(*) Work done as a member of GNAFA of CNR.
(**) Nella seduta del 14 febbraio 1981.

It is obvious that a strict solution is also a strong solution. From § III.4 [1] it can be deduced the following result:

PROPOSITION 1. *$u \in X$ is a strong solution of (1) if and only if we have:*

$$u(t) = e^{\Lambda t} x + \int_0^t e^{\Lambda(t-s)} \varphi(s, u(s))\, ds \qquad 0 \leq t \leq T. \tag{2}$$

DEFINITION 3. A function $u \in X$, solution of (2) is called a *mild solution* of (1).

DEFINITION 4. A function $u \in X$ such that $u(0) = x$ and $u'(t)$, $\Lambda u(t)$ exist continuous and verify (1_1) for $0 < t \leq T$ is called a *classical solution* of (1).

2. THE LINEAR CASE

Let us first consider the case in which φ does not depend on u:

$$\begin{cases} u'(t) = \Lambda u(t) + f(t) \\ u(0) = x. \end{cases} \tag{3}$$

We will use the notation and definitions of [5] for the spaces of hölder continuous functions C^θ, little hölder continuous functions h^θ and for the intermediate spaces $D_\Lambda(\theta)$ and $D_\Lambda(\theta, \infty)$.

For the mild solution of (3):

$$u(t) = e^{\Lambda t} x + \int_0^t e^{\Lambda(t-s)} f(s)\, ds \qquad 0 \leq t \leq T \tag{4}$$

we have the following result:

THEOREM 1. *Let $f \in C(0, T; E)$.*

(*i*) If $x \in E$, then $u \in h^\theta(\varepsilon, T; E) \cap C(\varepsilon, T; D_\Lambda(\theta))$ $\forall \varepsilon \in]0, T[$, $\forall \theta \in]0, 1[$

(*ii*) If $x \in D_\Lambda(\bar\theta, \infty)$ with $\bar\theta \in]0, 1[$ then $u \in C^{\bar\theta}(0, T; E)$

(*iii*) If $x \in D_\Lambda(\bar\theta)$ with $\bar\theta \in]0, 1[$, then $u \in h^{\bar\theta}(0, T; E) \cap C(0, T; D_\Lambda(\bar\theta))$.

The proof follows from Theorems 1,4 and Proposition 2 of [5]. See also the remark at the end of this paper.

The following theorems give conditions for the mild solution u of (3) to be classical or strict.

THEOREM 2. *Let* $f \in C^{\overline{\theta}}(0, T; E)$ with $\overline{\theta} \in]0, 1[$.

(*i*) If $x \in E$, then u is a classical solution of (3) and u', $Au \in C^{\overline{\theta}}(\varepsilon, T; E)$, $\forall \varepsilon \in]0, T[$.

(*ii*) If $x \in D_A$, then is a strict solution of (3).

(*iii*) If $x \in D_A$ and $Ax + f(0) \in D_A(\overline{\theta}, \infty)$, then u is a strict solution of (3) and u', $Au \in C^{\overline{\theta}}(0, T; E)$.

Proof. (*iii*) is proved in Theorem 7 of [5]. (*i*) and (*ii*) can be established by means of the decomposition (12) of [5] and by applying Theorem 2 and Proposition 2 of [5].

THEOREM 3. *Let* $f \in h^{\overline{\theta}}(0, T; E)$ *with* $\overline{\theta} \in]0, 1[$.

(*i*) If $x \in E$, then u is a classical solution of (3) and u', $Au \in h^{\overline{\theta}}(\varepsilon, T; E)$, $\forall \varepsilon \in]0, T[$.

(*ii*) If $x \in D_A$, then u is a strict solution of (3).

(*iii*) If $x \in D_A$ and $Ax + f(0) \in D_A(\overline{\theta})$, then u is a strict solution of (3) and u', $Au \in h^{\overline{\theta}}(0, T; E)$.

The proof is similar to that of the preceding theorem.

THEOREM 4. *Let* $f \in C(0, T; D(\overline{\theta}))$ *with* $\overline{\theta} \in]0, 1[$.

(*i*) If $x \in E$, then u is a classical solution of (3) and u', $Au \in C(\varepsilon, T; D_A(\overline{\theta}))$, $\forall \varepsilon \in]0, T[$.

(*ii*) If $x \in D_A$, then u is a strict solution of (3).

(*iii*) If $x \in D_A$ and $Ax \in D_A(\overline{\theta})$, then u is a strict solution of (3) and u', $Au \in C(0, T; D_A(\overline{\theta}))$.

Proof. It is sufficient to apply Theorem 6 of [5].

3. THE SEMILINEAR CASE

Several conditions are known for the existence of a mild solution of the semilinear equation (1): see for instance [4]. In this section we shall give some properties of the mild solutions of (1) and conditions for their regularity by using the results of the preceding section.

In what follows we shall suppose that $u \in C(0, T; E)$ is a given mild solution of (1) and that f is defined by

$$f(t) = \varphi(t, u(t)) \qquad 0 \le t \le T. \tag{5}$$

Hence u is a mild solution of the linear equation (3) so that we must have:

$$u(t) = e^{At} x + \int_0^t e^{A(t-s)} f(s) \, ds \qquad 0 \le t \le T. \tag{6}$$

THEOREM 5. *Let* $\varphi \in C([0, T] \times E ; E)$.

(*i*) If $x \in E$, then $u \in h^{\theta}(\varepsilon, T ; E) \cap C(\varepsilon, T ; D_A(\theta))$, $\forall \varepsilon \in]0, T[$, $\forall \theta \in]0, 1[$.

(*ii*) If $x \in D_A(\bar{\theta}, \infty)$ with $\bar{\theta} \in]0, 1[$, then $u \in C^{\bar{\theta}}(0, T ; E)$.

(*iii*) If $x \in D_A(\bar{\theta})$ with $\bar{\theta} \in]0, 1[$, then $u \in h^{\bar{\theta}}(0, T ; E) \cap \cap C(0, T ; D_A(\bar{\theta}))$.

Proof. As u verifies (6) and f is continuous, the conclusion is a consequence of Theorem 1.

Let us give now some conditions which guarantee that a mild solution u of (1) is classical or strict.

THEOREM 6. *Let* $\varphi \in C^{\alpha}_{loc}([0, T] \times E ; E)$ *with* $0 < \alpha \leq 1$ *i. e. for each* $x_0 \in E$ *let* $P(x_0, r_0) = \{x \in E ; \|x_0 - x\| \leq r_0\}$ *exist such that* $\varphi \in C^{\alpha}([0, T] \times \times P(x_0, r_0) ; E)$.

(*i*) If $x \in E$, then u is a classical solution of (1) and u', $Au \in C^{\beta}(\varepsilon, T ; E)$ $\forall \beta \in]0, \alpha[$, $\forall \varepsilon \in]0, T[$

(*ii*) If $x \in D_A$, then u is a strict solution of (1)

(*iii*) If $x \in D_A$ and $Ax + \varphi(0, x) \in D_A(\bar{\theta}, \infty)$ with $\bar{\theta} < \alpha$, then u is a strict solution of (1) and u', $Au \in C^{\bar{\theta}}(0, T ; E)$.

Proof. From Theorem 5 we have for each $\varepsilon \in]0, T[$ and $\theta \in]0, 1[$, $u \in C^{\theta}(\varepsilon, T ; E)$: hence $f \in C^{\beta}(\varepsilon, T ; E)$ for each $\beta \in]0, \alpha[$. If we consider problem $u' = Au + f$ for $t \geq \varepsilon$ then (*i*) follows from (*i*) of Theorem 2. If $x \in D_A$, from (*ii*) of Theorem 5 we get $u \in C^{\theta}(0, T ; E)$ for each $\theta \in]0, 1[$: hence $f \in C^{\beta}(0, T ; E)$ $\forall \beta \in]0, \alpha[$. Now (*ii*) can be deduced from (*ii*) of Theorem 2. If moreover $Ax + \varphi(0, x) \in D_A(\bar{\theta}, \infty)$, taking $\beta = \bar{\theta}$ we get (*iii*) from (*iii*) of Theorem 2.

THEOREM 7. *Let* $\varphi \in h^{\alpha}_{loc}([0, T] \times E ; E)$ *with* $0 < \alpha \leq 1$, $x \in D_A$ *and* $Ax + \varphi(0, x) \in D_A(\theta)$ *with* $0 < \theta < \alpha \leq 1$ *then u is a strict solution of* (1) *and* u', $Au \in h^{\theta}(0, T ; E)$.

Proof. From (*iii*) of Theorem 5, we have $u \in h^{\theta}(0, T ; E)$ for each $\theta \in]0, 1[$: hence $f \in h^{\beta}(0, T ; E)$ $\forall \beta \in]0, \alpha[$; now the conclusion follows from (*iii*) of Theorem 3.

Remark 2. It is easy to derive results similar to that of Theorems 6 and 7 if we suppose that $\varphi(t, u)$ is holder continuous with different exponents with respect to t and u: these results would be a generalization of Theorems 2 and 3.

THEOREM 8. *Let* $\varphi \in C([0, T] \times E ; E) \cap C([0, T] \times D_A(\theta_1) ; D_A(\theta_2))$ *with* $\theta_1, \theta_2 \in]0, 1[$.

(*i*) If $x \in E$, then u is a classical solution of (1) and u', $Au \in C(\varepsilon, T ; D_A(\theta_2))$, $\forall \varepsilon \in]0, T[$.

(*ii*) If $x \in D_A$, then u is a strict solution of (1).

(*iii*) If $x \in D_A$ and $Ax \in D_A(\theta_2)$, then u is a strict solution of (1) and $u', Au \in C(0, T; D_A(\theta_2))$.

Proof. From (*i*) of Theorem 5 we have $u \in C(\varepsilon, T; D_A(\theta_1))$, $\forall \varepsilon \in]0, T[$; hence $f \in C(\varepsilon, T; D_A(\theta_2))$: by applying (*i*) of Theorem 4 to the equation $u' = Au + f$ for $t \geq \varepsilon$ we get (*i*). If $x \in D_A$, from (*iii*) of Theorem 5 we obtain $u \in C(0, T; D_A(\theta_1))$ hence $f \in C(0, T; D_A(\theta_2))$ and (*ii*) follows from (*ii*) of Theorem 4. If moreover $Ax \in D_A(\theta_2)$ we can use (*iii*) of Theorem 4 to get (*iii*).

4. THE NON CONTINUOUS CASE

In this section we suppose that $x \to \varphi(t, x)$ is continuous only from $D_A(\theta_1)$ to E: more precisely we assume that $\varphi \in C([0, T] \times D_A(\theta_1); E)$.

The definition of strict solution of (1) remains the same but to the definitions of mild and classical solution we must add the condition $u \in C(0, T; D_A(\theta_1))$.

In [6] conditions for the existence, uniqueness and regularity of the mild solution of (1) are given; we want to prove more properties of the mild solutions and give a new condition for their regularity.

In what follows we shall suppose that $u \in C(0, T; D_A(\theta_1))$ is a given mild solution of (1) and shall define f as in (5); so that $f \in C(0, T; E)$ and we can derive from Theorem 1 the following.

THEOREM 9. *Let* $\varphi \in C([0, T] \times D_A(\theta_1); E)$. *Then* (*i*)-(*iii*) *of theorem 5 hold.*

To prove the existence of a classical solution or of a strict solution we need the following property of the mild solution:

THEOREM 10. *Let* $\varphi \in C([0, T] \times D_A(\theta_1); E)$.

(*i*) If $x \in E$, then $u \in h^\alpha(\varepsilon, T; D_A(\theta))$ for each $0 < \alpha < 1 - \theta < 1$ and $\varepsilon \in]0, T[$

(*ii*) If $x \in D_A(\alpha_1 + \theta_1)$ with $0 < \alpha_1 < 1 - \theta_1 < 1$, then $u \in h^{\alpha_1}(0, T; D_A(\theta_1))$.

Proof. From Theorem 5 of [5] we deduce (*i*); then (*ii*) can be proved by using Theorem 2. 10 of [3].

It is possible to demonstrate analogously to Theorems 6 and 7 the following regularity results for equation (1):

THEOREM 11. *Let* $\varphi \in C^\alpha_{loc}([0, T] \times D_A(\theta_1); E)$ *with* $0 < \alpha \leq 1$.

(*i*) If $x \in E$, then u is a classical solution of (1) and $u', Au \in C^\beta(\varepsilon, T; E)$ for each $\beta < \alpha(1 - \theta_1)$ and $\varepsilon \in]0, T[$.

(*ii*) If $x \in D_\Lambda$, then u is a strict solution of (1).

(*iii*) If $x \in D_\Lambda$ and $\Lambda x + \varphi(0, x) \in D_\Lambda(\bar\theta, \infty)$ with $\bar\theta < \alpha(1 - \theta_1)$, then u is a strict solution of (1) and u', $Au \in C^{\bar\theta}(0, T; E)$.

THEOREM 12. *Let* $\varphi \in h^\alpha_{loc}([0, T] \times D_\Lambda(\theta_1); E)$ with $0 < \alpha \leq 1$, $x \in D_\Lambda$ and $\Lambda x + \varphi(0, x) \in D_\Lambda(\bar\theta)$ with $\bar\theta < \alpha(1 - \theta_1)$; then u is a strict solution of (1) and u', $Au \in h^{\bar\theta}(0, T; E)$.

Part of the following theorem was proved in [6]:

THEOREM 13. *Let* $\varphi \in C([0, T] \times D_\Lambda(\theta_1); D_\Lambda(\theta_2))$ *with* $\theta_1, \theta_2 \in]0, 1[$. *Then* (*i*)–(*iii*) *of theorem 8 hold.*

The proof is obtained from theorem 9.

Remark 3. There is a certain lack of simmetry between $C(0, T; D_\Lambda(\theta))$ and $C(0, T; D_\Lambda(\theta, \infty))$: see Theorems 1,5 and the theorems which should correspond to Theorems 4 and 8. This is due to the that fact $C(0, T; D_\Lambda(\theta, \infty))$ is not in general an intermediate space between $C(0, T; D_\Lambda)$ and $C(0, T; E)$ in the sense of Definition 6 of [5]: so that lines 3 and 4 of page 385 (and consequently (*ii'*) of Theorem 4. 7) of [2] should be changed by replacing $C(0, T; D_\Lambda(\theta, \infty))$ by another space which will be specified in the detailed version of this note, in which additional regularity results will be given.

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