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Real zeros of general L-functions

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Teoria dei numeri. — *Real zeros of general L-functions.* Nota di ALBERTO PERELLI e GIUSEPPE PUGLISI, presentata (*) dal Socio G. Zappa.

RIASSUNTO. — In questo lavoro vengono studiati gli zeri reali di una classe di serie di Dirichlet, che generalizzano le funzioni L (s, χ), definite in [8].

Combinando le tecniche elementari di Pintz [9] con alcuni metodi analitici si ottiene l'estensione dei classici teoremi di Hecke e Siegel.

1. INTRODUCTION AND STATEMENT OF RESULTS

Siegel zeros of Dirichlet L-functions formed with real primitive characters are of importance in various arithmetic problems, as the estimation of the error term in the Prime Number Theorem for arithmetic progressions, or the determination of all quadratic fields with a given class number. For an account of this subject, in particular for the theorems of Hecke [4], Landau [4], Page [7], Siegel [10] and their consequences, we refer to Davenport's book [2].

Elementary proofs of these theorems have been recently given by Pintz [9], while some analogues for zeta-functions associated to automorphic representations of $GL_2(\mathbf{A})$, where $\mathbf{A}$ is the adele ring of the rationals, and for Hecke L-functions, have been obtained by Moreno [6] and Fogels [3], respectively.

The aim of the present paper is to prove similar results for a wide class of L-functions, namely for the " real " general L-functions, as defined by the axioms (A1)—(A5) in Perelli [8]. Here we note that the remarks *a*) and *b*) in [8] apply to real general L-functions as well.

Special cases of real general L-functions are the Dirichlet L-functions L (s, χ) with real primitive χ, the Hecke L-functions (see Lang [5]), and the zeta-functions associated with cusp forms for $SL_2(\mathbf{Z})$.

Unlike the case of L (s, χ) (see Pintz [9]), complex zero-free regions are required for the detection of the real zeros of L $(s, \mathscr{A})$. To this purpose, the known methods make use of suitable Euler products related to L $(s, \mathscr{A})$. We are thus led to make the following assumption in the sequel:

Hypothesis S—T. *Let* L $(s, \mathscr{A})$ *be a real general* L*-function. If*

$$\mathscr{A} \otimes \mathscr{A} = (A_p \otimes A_p)_{p \in \mathscr{P}},$$

then the Euler product L $(s, \mathscr{A} \otimes \mathscr{A})$ *is a general* L*-function with simple pole at* $s = 1$.

(*) Nella seduta del 14 febbraio 1981.

The parameters of $L(s, \mathscr{A} \otimes \mathscr{A})$ depend explicitly on those of $L(s, \mathscr{A})$. In particular, Q_0, the main parameter of $L(s, \mathscr{A} \otimes \mathscr{A})$, is a function of Q. We let

$$\overline{Q} = \max(Q, Q_0).$$

We have the following

THEOREM A. *Let* $L(s, \mathscr{A})$ *be a real general* L*-function satisfying Hypothesis* S—T. *Then* $L(s, \mathscr{A}) \neq 0$ *for*

$$\sigma \geq 1 - \frac{c_1}{\log[\overline{Q}(|t| + 2)]} \qquad (c_1 > 0)$$

except possibly for simple real zero $\beta_0 < 1$.

For the proof of Theorem A and for some remarks on Hypothesis S—T, see Perelli [8].

The generalization of Landau's theorem for $L(s, \mathscr{A})$ is contained in Theorem A.

Using analytic tools and some ideas of Pintz [9], we can also extend the theorems of Hecke and Siegel, in the following forms.

THEOREM 1. *If* $L(s, \mathscr{A})$ *satisfies Hypothesis* S—T *and* $L(\sigma, \mathscr{A}) \neq 0$ *for* $\beta \leq \sigma \leq 1$, *with* $0 < 1 - \beta < c_2/\log \overline{Q}$, $c_2 > 0$ *independent of* $\overline{Q}$, *then*

$$L(1, \mathscr{A}) \gg (1 - \beta)^M.$$

THEOREM 2. *If* $L(s, \mathscr{A})$ *satisfies Hypothesis* S—T *and* $L(\beta_0, \mathscr{A}) = 0$, $\beta_0 < 1$, *then, for any* $\varepsilon > 0$,

$$1 - \beta_0 \gg_\varepsilon \overline{Q}^{-\varepsilon},$$

where the dependence on ε *of the implied constants is ineffective.*

On combining Theorems 1 and 2 we get as usual

THEOREM 3. *If* $L(s, \mathscr{A})$ *satisfies Hypothesis* S—T *then, for any* $\varepsilon > 0$,

$$L(1, \mathscr{A}) \gg_\varepsilon \overline{Q}^{-\varepsilon}.$$

2. Basic formula

Let

$$L(s, \mathscr{A}) = \prod_p \prod_{j=1}^{M} (1 - \chi_j(p) p^{-s}),$$

$$\tilde{M} = \begin{cases} M & \text{if } M \text{ is odd} \\ M + 1 & \text{if } M \text{ is even.} \end{cases}$$

We define

$$F_{\tilde{M}}(s) = \zeta(s)^{\tilde{M}} L(s, \mathscr{A}) = \sum_{n=1}^{\infty} a_{\tilde{M}}(n) n^{-s} \qquad (\sigma > 1).$$

From the assumptions on $\chi_j(p)$ it follows that

$$a_{\tilde{M}}(n) \geq 0\,, \tag{2.1}$$

as it is clear from the Dirichlet series for $\log F_M(s)$. We use Perron's formula (for h large enough):

$$H_h(x) = \frac{1}{h!} \sum_{n \leq x} a_{\tilde{M}}(n) \log^h \frac{x}{n} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F_{\tilde{M}}(s) \frac{x^s}{s^{h+1}}\, ds\,.$$

Shifting the line of integration to $\sigma = 1 - \varepsilon'$, with a small $\varepsilon' > 0$ independent of $\overline{Q}$, and computing the residue of the integrand at $s = 1$, we obtain the asymptotic formula for $H_h(x)$. Then (2.1) allows us to derive the asymptotic formula for $H_0(x)$, by well known tauberian arguments. By partial summation we finally obtain the following generalization of formula (3.12) of Pintz [9, VIII]:

$$\sum_{n \leq x} a_{\tilde{M}}(n)\, n^{-s} = F_{\tilde{M}}(s) - \frac{\lambda_0\, s x^{1-s}}{(s-1)^{\tilde{M}}} L(1, \mathscr{A}) + \sum_{i=1}^{\tilde{M}-2} \frac{\lambda_i(x)\, s x^{1-s}}{(s-1)^{\tilde{M}-i}} \tag{2.2}$$

$$+ \frac{\lambda_{\tilde{M}-1}(x)\, x^{1-s}}{s-1} + \lambda_{\tilde{M}}(x)\, x^{1-s} + 0\left(|s|\, K x^{1-\sigma-\varepsilon}\right),$$

where $\lambda_0 > 0$, $\lambda_i(x)$ $(i = 1, \cdots, \tilde{M})$ is a linear combination with real coefficients of $L^{(j)}(1, \mathscr{A}) \log^l x$ $(0 \leq j + l \leq \min(i, \tilde{M} - 1))$, $K = Q + \sum_{j=0}^{\tilde{M}-1} |L^{(j)}(1, \mathscr{A})|$, and $\varepsilon > 0$ is independent of $\overline{Q}$.

3. ESTIMATION OF $L^{(j)}/L$

From theorem 4.1 in [1] it follows by partial summation that

$$L^{(j)}(\sigma, \mathscr{A}) \ll (\log Q)^{M+j} \qquad (j = 0, 1, 2, \cdots) \tag{3.1}$$

for $0 < 1 - \sigma \ll (\log Q)^{-1}$.

LEMMA. *If there exists*

$$\beta_0 \geq 1 - \frac{c_1}{\log(2\overline{Q})}$$

such that $L(\beta_0, \mathscr{A}) = 0$, *then*

$$\frac{L'}{L}(s, \mathscr{A}) = \frac{1}{s - \beta_0} + 0(\log \overline{Q})$$

for

$$\sigma \geq 1 - \frac{c_1}{4 \log(3\overline{Q})}, \qquad |t| \leq 1.$$

Proof. We use a well known method of Landau (see [11], § 3.9). $L(s, \mathscr{A})$ is real for real s, so we may obviously assume $t \geq 0$. We consider the entire function

$$f(s) = \frac{L(s, \mathscr{A})}{s - \beta_0} .$$

By Theorem A, $f(s)$ does not vanish for

$$\left\{ \begin{array}{l} \sigma \geq 1 - \dfrac{c_1}{\log (3 \overline{Q})} \\ t \leq 1 . \end{array} \right. \tag{3.2}$$

Let

$$\sigma_0 = 1 + \frac{c_1}{2 \log (3 \overline{Q})} .$$

By (3.1) we have

$$f(s) = \sum_{j=1}^{\infty} L^{(j)}(\beta_0, \mathscr{A}) \frac{(s - \beta_0)^{j-1}}{j!} \ll (\log \overline{Q})^{M+1}$$

for

$$|s - \beta_0| \ll \frac{1}{\log \overline{Q}} ,$$

while

$$f(s) \ll \{\overline{Q}(|t| + 1)\}^{c_3}$$

for

$$\tfrac{1}{2} \leq \sigma \leq 2 \quad , \quad |\sigma - \beta_0| \gg \frac{1}{\log \overline{Q}} .$$

By the Euler product for $L(s, \mathscr{A})$ it follows that

$$L(\sigma_0, \mathscr{A})^{-1} \ll (\sigma_0 - 1)^{-M}.$$

Hence, in the circle $|s - \sigma_0| \leq \frac{1}{2}$,

$$\frac{f(s)}{f(\sigma_0)} \ll \overline{Q}^{c_4} . \tag{3.3}$$

Further

$$\frac{L'}{L}(\sigma_0, \mathscr{A}) - \frac{1}{\sigma_0 - \beta_0} \ll \frac{1}{\sigma_0 - 1} + \frac{1}{\sigma_0 - \beta_0} \ll \log \overline{Q} . \tag{3.4}$$

By (3.2), (3.3) and (3.4), the lemma is a straightforward consequence of lemma γ in [11], p. 50.

From Borel–Carathéodory's theorem we have

$$\left(\frac{L'}{L}\right)^{(i)}(1, \mathscr{A}) = \frac{(-1)^i i!}{(1 - \beta_0)^{i+1}} + 0((\log \overline{Q})^{i+1}) \qquad (i = 0, 1, \cdots). \tag{3.5}$$

Writing $L^{(j)}/L$ in terms of the derivatives of L'/L, we obtain by (3.5)

$$\frac{L^{(j)}}{L}(1, \mathscr{A}) \ll \frac{\log \overline{Q}}{(1-\beta_0)^{j-1}} \qquad (j \geqq 2). \tag{3.6}$$

4. Proof of Theorem 1

From Theorem A and (2.1) it follows that $F_{\tilde{M}}(\sigma) < 0$ for $\beta \leq \sigma < 1$. Hence, by (2.2),

$$\sum_{n \leq x} a_{\tilde{M}}(n)\, n^{-\beta} \leq \frac{\beta x^{1-\beta}}{(1-\beta)^{\tilde{M}}} L(1, \mathscr{A}) \left\{ \lambda_0 + \sum_{i=1}^{\tilde{M}-2} (-1)^{i-1} \frac{\lambda_i(x)(1-\beta)^i}{L(i, \mathscr{A})} - \right.$$
$$\left. - \frac{\lambda_{\tilde{M}-1}(x)(1-\beta)^{\tilde{M}-1}}{\beta_1 L(1, \mathscr{A})} + \frac{\lambda_{\tilde{M}}(x)(1-\beta)^{\tilde{M}}}{\beta L(1, \mathscr{A})} + 0\left(\frac{(1-\beta)^{\tilde{M}} K x^{-\varepsilon}}{L(1, \mathscr{A})}\right)\right\}. \tag{4.1}$$

We choose $x = \exp \dfrac{1}{1-\beta}$ in (4.1). From (2.1), (3.5), (3.6) and the definition of $\lambda_i(x)$, we obtain

$$1 \leq \sum_{n \leq \exp(1/1-\beta)} a_{\tilde{M}}(n)\, n^{-\beta} \ll \frac{L(1, \mathscr{A})}{(1-\beta)^{\tilde{M}}}. \tag{4.2}$$

For even M, we have by the definition of $\tilde{M}$

$$a_{\tilde{M}} = a_M * 1,$$

with $a_M(n) \geq 0$. Therefore

$$\sum_{n \leq \exp(1/1-p)} a_{\tilde{M}}(n)\, n^{-\beta} = \sum_{d \leq \exp(1/1-\beta)} a_M(d)\, d^{-\beta} \sum_{k \leq d^{-1} \exp(1/1-\beta)} k^{-\beta} \geq$$
$$\geq \sum_{k \leq \exp(1/1-\beta)} k^{-\beta} > \frac{1}{1-\beta}.$$

From (4.2) we have, in any case,

$$L(1, \mathscr{A}) \gg (1-\beta)^{M}.$$

5. Proof of Theorem 2

We may obviously assume that there exists $L_1(s, \mathscr{A})$ satisfying $L_1(\gamma_0, \mathscr{A}) = 0$, with

$$1 - \beta_0 < 1 - \gamma_0 < \varepsilon. \tag{5.1}$$

Denoting by M and M_1 the number of Euler factors of $L(s, \mathscr{A})$ and $L_1(s, \mathscr{A})$ respectively, it is clear that $M_2 = M + M_1$ is the same number for $L(s, \mathscr{A}) L_1(s, \mathscr{A})$. We apply (2.2) to the function

$$\zeta(s)^{\tilde{M}_2} L(s, \mathscr{A}) L_1(s, \mathscr{A}) = \sum_{n=1}^{\infty} a_{\tilde{M}_2}(n) n^{-s}$$

at the point $s = \beta_0$ and $s = \gamma_0$. Since, by (5.1),

$$\sum_{n \leq x} a_{\tilde{M}_2}(n) n^{-\beta_0} \leq \sum_{n \leq x} a_{\tilde{M}_2}(n) n^{-\gamma_0},$$

we obtain

$$\frac{\beta_0 x^{1-\beta_0}}{(1-\beta_0)^{\tilde{M}_2}} (1 + G(\beta_0, x)) \leq \frac{\gamma_0 x^{1-\gamma_0}}{(1-\gamma_0)^{\tilde{M}_2}} (1 + G(\gamma_0, x)), \tag{5.2}$$

where, choosing

$$x = (\overline{Q}\,\overline{Q}_1)^{c_5},$$

we have

$$\begin{cases} 1 + G(\beta_0, x) \gg 1, \\ 1 + G(\gamma_0, x) \ll (1-\beta_0)^{-(\tilde{M}_2-1)}, \end{cases} \tag{5.3}$$

by the same argument as in the proof of Theorem 1.

From (5.2) and (5.3) it follows that

$$1 - \beta_0 \gg x^{-(\beta_0-\gamma_0)} \geq \overline{Q}^{-\varepsilon}.$$

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