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A remark on the asymmetry of convolution operators

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Analisi matematica. — *A remark on the asymmetry of convolution operators.* Nota di SAVERIO GIULINI, presentata (*) dal Socio L. AMERIO.

ABSTRACT. — A convolution operator, bounded on $L^q(\mathbb{R}^n)$, is bounded on $L^p(\mathbb{R}^n)$, with the same operator norm, if p and q are conjugate exponents. It is well known that this fact is false if we replace $\mathbb{R}^n$ with a general non-commutative locally compact group G. In this paper we give a simple construction of a convolution operator on a suitable compact group G, wich is bounded on $L^q(G)$ for every $q \in [2, \infty)$ and is unbounded on $L^p(G)$ if $p \in [1,2)$.

KEY WORDS: Non-commutative groups; Convolution operators; Asymmetry.

RIASSUNTO. — *Una osservazione sulla asimmetria degli operatori di convoluzione.* È noto che un convolutore limitato su $L^q(\mathbb{R}^n)$ è anche limitato su $L^p(\mathbb{R}^n)$ se q e p sono esponenti coniugati: inoltre si ha eguaglianza delle norme. Questo fatto non è più vero se a $\mathbb{R}^n$ si sostituisce un generico gruppo G localmente compatto non commutativo. Ciò è stato dimostrato tempo fa per particolari (intervalli di) valori di q e p. In questo lavoro si costruisce un gruppo compatto G per il quale esiste una famiglia di convolutori limitati su $L^q(G)$ *per ogni* $q \in [2, \infty)$, i quali non sono limitati per nessun $p \in [1,2)$.

Let G be a locally compact group. We denote by $\mathrm{Conv}_p(G)$ the space of L^p left convolution operators on G and by $|||\cdot|||_p$ the usual norm in $\mathrm{Conv}_p(G)$. If G is abelian it is well known that for every T in $\mathrm{Conv}_p(G)$ we have $|||T|||_p = |||T|||_{p'}$ if $p'^{-1} + p^{-1} = 1$. This equality is in general false if G is non-abelian. The first counterexample was produced by D. Oberlin [7]: he proved the existence of a left convolution operator T in the dihedral group D_4 such that $|||T|||_4 < |||T|||_{4/3}$. Subsequently C. Herz [3] discovered a much more general method to compute convolutor norms and proved that for every finite non-commutative group and for each p, $1 < p < 2$, there exists a left convolution operator T, which depends on p, such that $|||T|||_{p'} < |||T|||_p$. These facts combined with the Uniform Boundedness Principle allows us to deduce that $\mathrm{Conv}_p(G) \neq \mathrm{Conv}_{p'}(G)$ for suitable G.

A natural problem is to look for a left convolutor T which does not depend on p and which is unbounded on L^p if $1 \leqq p < 2$ and bounded on L^q if $2 \leqq q < \infty$. In the case of non-compact groups, A. M. Mantero produced examples which satisfy the previous asymmetry condition for the Heisenberg group ([5], [6]). Moreover M. Baronti and G. Foresti [1] proved the existence of a compact group G and a convolution operator T such that $T \in \mathrm{Conv}_q(G)$ for $2 \leq q \leq 4$ but $T \notin \mathrm{Conv}_p(G)$ if $1 \leq p < 2$.

In this paper we use Oberlin's example to give a very simple construction of a compact group G and of a family of left convolutors $\{T_\lambda\}$ such that $T_\lambda \notin \mathrm{Conv}_p(G)$ if $1 \leq p < 2$, but $T_\lambda \in \mathrm{Conv}_q(G)$ if $2 \leq q < \infty$.

We recall a few facts about dihedral groups (see [4], (27.61) *d*). The dihedral group D_m is the semidirect product $\mathbb{Z}_m \ltimes \mathbb{Z}_2$ $(m > 2)$. A complete list of its unitary irreducible representations is the following (up to equivalence):

dimension 1: $\mathbf{1}: (\pm 1, \mathrm{k}) \to 1$, $\psi_1: (\pm 1, \mathrm{k}) \to (\pm 1)^{\mathrm{k}}$, and, if m is even,

$$\psi_2: (\pm 1, \mathrm{k}) \to (-1)^{\mathrm{k}}, \qquad \psi_3: (\pm 1, \mathrm{k}) \to \pm(-1)^{\mathrm{k}};$$

(*) Nella seduta del 14 gennaio 1989.

dimension 2:

$$U_n: \begin{cases} (1,k) \to \begin{bmatrix} \alpha^{nk} & 0 \\ 0 & \alpha^{-nk} \end{bmatrix} \\ (-1,k) \to \begin{bmatrix} 0 & \alpha^{-nk} \\ \alpha^{-nk} & 0 \end{bmatrix} \end{cases}$$

where $\alpha = \exp(2\pi i/m)$, and $n = 1, 2, \dots, [(m-1)/2]$.

Following Herz notation [3] let φ be the irreducible subcharacter of D_m defined by

$$\varphi(x) = \begin{cases} \alpha^k & \text{if } x = (1,k), \\ 0 & \text{if } x = (-1,k). \end{cases}$$

The smallest closed subspace $E(\varphi)$ of $L^2(D_m)$ which contains φ and is stable for right translation is the two dimensional complex vector space of functions on D_m such that

$$f(\pm 1, k) = \alpha^k f(\pm 1, 0).$$

If $f \in E(\varphi)$, its Fourier treansform is supported on $\{U_1\}$ and

$$f^\wedge(U_1) = \frac{1}{2}\begin{bmatrix} f(1,0) & f(-1,0) \\ 0 & 0 \end{bmatrix}.$$

We put $E^*(\varphi) = \{f^* : f \in E(\varphi)\}$ where $\overline{f^*}(x) = f(x^{-1})$. Obviously $E^*(\varphi)$ is the smallest closed subspace of $L^2(D_m)$ which contains φ and is stable for left translation. C. Herz [3] proved that if $K \in E^*(\varphi)$ then

$$(1) \qquad |||K*|||_p = 2^{-1/p}\|K\|_p = 2^{-1}(|K(1,0)|^p + |K(-1,0)|^p)^{1/p} \qquad 1 \le p \le 2$$

(here $K*$ denotes the convolutor $g \to K*g$ and the meausure on D_m is normalized so that the total mass is equal to 1). Moreover

$$(2) \qquad |||K*|||_p = \sup_{g \in E(\varphi)} \frac{\|K*g\|_p}{\|g\|_p} \qquad \text{for every } p.$$

Theorem. There exist a non-commutative compact group G and a convolution operator T such that $T \in \mathrm{Conv}_p(G)$ if $p \in [2, \infty)$ but $T \notin \mathrm{Conv}_q(G)$ if $q \in [1, 2)$.

Proof. We consider the dihedral group D_4 and we choose $K \in E^*(\varphi)$ such that $(K_\lambda)^\wedge(U_1) = \begin{bmatrix} 1 & 0 \\ \lambda & 0 \end{bmatrix}$. We define the linear operator $L_{\lambda,\mu}: \mathbf{C}^2 \to \mathbf{C}^4$, $\lambda, \mu \in \mathbf{R}$, by

$$L_{\lambda,\mu}(z_1, z_2) = (\mu z_1 + \lambda z_2, \mu z_1 - \lambda z_2, \mu z_2 + \lambda z_1, \mu z_2 - \lambda z_1).$$

For $g \in E(\varphi)$ we put $g(1,0) = c_1$ and $g(-1,0) = c_2$. Then

$$(3) \qquad \left(\frac{\|K*g\|_p}{\|g\|_p}\right)^p = \frac{|c_1 + \lambda c_2|^p + |c_1 - \lambda c_2|^p + |c_2 + \lambda c_1|^p + |c_2 - \lambda c_1|^p}{2(|c_1|^p + |c_2|^p)} = \frac{1}{2}\left(\frac{\|L_{\lambda,1}(c_1, c_2)\|_{l^p}}{\|(c_1, c_2)\|_{l^p}}\right)^p.$$

Since

$$|\mu x \pm \lambda y|^2 = \mu^2 |x|^2 + \lambda^2 |y|^2 \pm 2\mu\lambda \operatorname{Re} x\bar{y} \leq \|(\mu+\lambda, \mu-\lambda\|_{l^\infty}^2 \|(x,y)\|_{l^\infty}^2$$

we have

$$\|L_{\lambda,\mu}(z_1,z_2)\|_{l^2} = \|(\mu+\lambda, \mu-\lambda)\|_{l^2} \|(z_1,z_2)\|_{l^2},$$
$$\|L_{\lambda,\mu}(z_1,z_2)\|_{l^\infty} \leq \|(\mu+\lambda, \mu-\lambda)\|_{l^\infty} \|(z_1,z_2)\|_{l^\infty}.$$

Therefore

$$\text{(4)} \qquad |||L_{\lambda,\mu}|||_2 = \|(\mu+\lambda, \mu-\lambda)\|_{l^2},$$
$$\text{(4')} \qquad |||L_{\lambda,\mu}|||_\infty = \|(\mu+\lambda, \mu-\lambda)\|_{l^\infty}.$$

We denote by $\mathbb{R}_p^n$ and $\mathbb{C}_p^n$ the vector space $\mathbb{R}^n$ and $\mathbf{C}^n$ equipped with the l^p-norm and by $\mathcal{L}(X, Y)$ the space of bounded linear operators between the normed spaces X and Y. We consider the operator $L\colon \mathbb{R}_p^2 \to \mathcal{L}(\mathbf{C}_p^2, \mathbf{C}_p^4)$ defined by

$$L(u,v) = L_{(u+v)/2,\,(u-v)/2}\,.$$

Since the norm in the interpolation space between $\mathcal{L}(\mathbf{C}_2^2, \mathbf{C}_2^4)$ and $\mathcal{L}(\mathbf{C}_\infty^2, \mathbf{C}_\infty^4)$ dominates that of $\mathcal{L}(\mathbf{C}_p^2, \mathbf{C}_p^4)$, $2 < p < \infty$, it follows from (4) and (4′) that

$$\text{(5)} \qquad |||L(u,v)|||_p \leq \|(u,v)\|_{l^p}, \qquad \text{for } p \in [2,\infty)$$

(actually the equality holds; here $|||\ |||_p$ denotes the norm in $\mathcal{L}(\mathbf{C}_p^2, \mathbf{C}_p^4)$). Finally if we put $u = 1+\lambda$ and $v = \lambda - 1$ we obtain

$$\text{(6)} \qquad |||K_\lambda *|||_p = \frac{1}{2}(|1+\lambda|^p + |1-\lambda|^p)^{1/p} \qquad \text{if } 2 \leq p < \infty.$$

Since $K_\lambda(1,0) = 2$ and $K_\lambda(-1,0) = 2\lambda$ it follows from (1) that

$$\text{(7)} \qquad |||K_\lambda *|||_{p'} = (1+|\lambda|^{p'})^{1/p'} \qquad \text{if } 2 \leq p < \infty.$$

Clarkson's inequality [2, Th. 2] states that

$$|1+\lambda|^p + |1-\lambda|^p \leq 2(1+|\lambda|^{p'})^{p/p'} \qquad \text{if } 2 \leq p < \infty.$$

and the equality holds if and only if $\lambda = 0, \pm 1$. Then

$$\text{(8)} \qquad |||K_\lambda *|||_p < |||K_\lambda *|||_{p'}$$

if $\lambda \neq 0, \pm 1$ and $2 < p < \infty$.

The proof ends as in [7]. It is enough to consider the Cartesian product G_n of n copies of D_4 and the function $H_{\lambda,n}$ on G_n defined by

$$(H_{\lambda,n})^\wedge(\pi_1, \ldots, \pi_n) = \hat{K}_\lambda(\pi_1) \otimes \ldots \otimes \hat{K}_\lambda(\pi_n)$$

for $(\pi_1, \ldots, \pi_n) \in \hat{G}_n$ and $\lambda \neq 0, \pm 1$. Then if $2 < p < \infty$

$$\frac{|||H_{\lambda,n}*|||_{p'}}{|||H_{\lambda,n}*|||_p} = \left(\frac{|||K_\lambda *|||_{p'}}{|||K_{\lambda,n}*|||_p}\right)^n \to \infty \qquad \text{as } n \to \infty.$$

The Uniform Boundedness Principle allows us to obtain the thesis for the group $G = \prod_{n=1}^\infty G_n$. □

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