Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Piermarco Cannarsa, Giuseppe Da Prato

The vanishing viscosity method in infinite dimensions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **83** (1989), n.1, p. 79–84.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1989_8_83_1_79_0>

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma bdim (Biblioteca Digitale Italiana di Matematica) SIMAI & UMI

http://www.bdim.eu/

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1989.

Atti Acc. Lincei Rend. fis.
(8), LXXXIII (1989), pp. 79-84

Analisi matematica. — *The vanishing viscosity method in infinite dimensions.* Nota di PIERMARCO CANNARSA e GIUSEPPE DA PRATO, presentata (*) dal Corrisp. A. AMBROSETTI.

ABSTRACT. — The vanishing viscosity method is adapted to the infinite dimensional case, by showing that the value function of a deterministic optimal control problem can be approximated by the solutions of suitable parabolic equations in Hilbert spaces.

KEY WORDS: Hamilton-Jacobi equations; Infinite dimensions; Viscosity solution; Optimal control.

RIASSUNTO. — *Il metodo della viscosità artificiale in dimensione infinita.* Si adatta il metodo della viscosità artificiale al caso infinito dimensionale, dimostrando che la funzione valore di un problema di controllo deterministico si può approssimare con soluzioni di equazioni paraboliche in spazi di Hilbert.

1. INTRODUCTION

It is well known (see for instance [15]) that, under suitable assumptions, the viscosity solution of the problem

$$-\frac{\partial v}{\partial t} + H(t, x, v_x) = 0 \qquad \text{in } [0,T] \times \boldsymbol{R}^n;\ v(T,x) = \phi(x) \tag{1.1}$$

can be obtained as uniform limit as $\varepsilon \downarrow 0$ of the solutions to the parabolic equations

$$-\frac{\partial v^\varepsilon}{\partial t} + H(t, x, v^\varepsilon_x) + \varepsilon \Delta v^\varepsilon = 0 \qquad \text{in } [0,T] \times \boldsymbol{R}^n;\ v^\varepsilon(T,x) = \phi(x)\,. \tag{1.1$_\varepsilon$}$$

When $H(t,x,\cdot)$ is convex, the Hamilton-Jacobi equation (1.1) is related to Optimal Control. Then, the convergence result above can be regarded as a way to transfer information from a PDE context to a variational problem.

Problem (1.1) has also been studied when $\boldsymbol{R}^n$ is replaced by a Hilbert space H or, more generally, by a Banach space X. In fact, in this case, it is related to Optimal Control problems for Distributed Parameter Systems. The theory of viscosity solutions has recently been adapted to the infinite dimensional case under fairly general assumptions for convex hamiltonians, see [5], [6], [7], [8]. Further extensions are given in [2], [3].

All this being said, we note that the methods used in the above papers to prove the existence of viscosity solutions are purely variational. Indeed, the equivalent of the vanishing viscosity method for the infinite dimensional case has not yet been developed.

The main difficulty in the application of this method is the fact that, in a Hilbert space H, there is no general theory available for partial differential equations of the form

$$-\frac{\partial v^\varepsilon}{\partial t} + H(t, x, v^\varepsilon_x) + \frac{\varepsilon}{2}\operatorname{Tr}(Q v^\varepsilon_{xx}) = 0 \qquad \text{in } [0,T] \times H;\ v^\varepsilon(T,x) = \phi(x)\,. \tag{1.2$_\varepsilon$}$$

Here Q is a positive nuclear operator in H and Tr denotes the trace.

(*) Nella seduta del 14 gennaio 1989.

Linear Parabolic Equations in infinite dimensions have been studied by using the abstract Wiener measure in the papers by [13], [9] and [16]. A nonlinear extension of these results has been obtained by [14]. Moreover, the existence of solutions in spaces of convex functions has been proved by [1].

A direct approach to $(1.2)_\varepsilon$ for non-convex data, which only uses Functional Analysis, has been introduced by [10]. This approach can only be applied to Hamilton-Jacobi equations with a special structure.

In the paper we utilize the results of [11] to solve problem $(1.2)_\varepsilon$ directly. Then, we show the convergence, as $\varepsilon \downarrow 0$, of V_ε to the viscosity solution V of the limit problem.

2. Notation and preliminaries

Let H a separable Hilbert space, Q a positive self-adjoint nuclear operator in H, A a self-adjoint negative operator which commutes with Q. We assume that there exists a complete orthonormal system $\{e_k\}$ and two sequences of real numbers $\{\lambda_k\}$ and $\{\mu_k\}$ such that:

$$Qe_k = \lambda_k e_k, \qquad Ae_k = \mu_k e_k; \qquad \lambda_k \geq 0, \ \mu_k < 0. \tag{2.1}$$

Let K another Hilbert space. We denote by $C_b(H,K)$ the set of all uniformly continuous and bounded mappings $\phi: H \to K$. Likewise, $C_b^1(H,K)$ is the set of all the mappings $\phi: H \to K$ which are Fréchet differentiable and uniformly continuous and bounded with their derivative.

We will also define a weaker notion of derivative, namely the *derivative in the direction Q*. We say that $f \in C_b(H, \boldsymbol{R})$ is differentiable in direction Q if

$$\lim_{h\to 0} (f(x+hQy) - f(x))/h =: \Gamma y$$

exists for all $y \in H$ and if Γ is a continuous linear functional on H. In this case, we denote by $Qf_x(x)$ the element of H which represents Γ, i.e. $\langle y, Qf_x(x)\rangle = \Gamma y$ for all $y \in H$. Let us also introduce the space

$$C_Q^1(H,\boldsymbol{R}) = \{f \in C_b(H,\boldsymbol{R}) : Qf_x \in C_b(H,H)\}.$$

Let now $F \in C_b^1(H,H)$ and consider the *State Equation*

$$y'(s) = Ay(s) + Q[F(y(s)) + u(s)], \qquad y(t) = x, \ t \leq s \leq T; \ u \in L^2(t,T;H) \tag{2.2}$$

which, by standard results, has a unique mild solution $y \in C([0,T];H)$. We recall that y is a *mild* solution of (2.2) if the following integral equation holds for all $s \in [t,T]$

$$y(s) = \exp[(s-t)A]x + \int_t^s \exp[(s-r)A]\, Q[F(r) + u(r)]\, dr.$$

We want to minimize the *Cost Functional*

$$J(t,x,u) = \int_t^T \left[g(y(s)) + \frac{1}{2}|u(s)|^2 \right] ds + \phi(y(T)) \tag{2.3}$$

where $g \in \mathrm{Lip}(H,\boldsymbol{R})$ and $\phi \in C_Q^1(H,\boldsymbol{R}) \cap \mathrm{Lip}(H,\boldsymbol{R})$. The Value Function of problem

(2.2), (2.3) is defined as follows

$$V(t,x) = \inf\{J(t,x,u);\ u \in L^2(t,T;H)\}\,. \tag{2.4}$$

Moreover, by [8] we conclude that V is the unique viscosity solution of the Hamilton-Jacobi equation

$$-\frac{\partial v}{\partial t} + \frac{1}{2}|Qv_x|^2 - \langle Ax + QF(x), v_x\rangle - g(x) = 0 \quad \text{in } [0,T]\times H;\ v(T,x) = \phi(x)\,. \tag{2.5}$$

Consider now a *Stochastic State Equation*

$$dy_\varepsilon(s) = \{Ay_\varepsilon(s) + Q[F(y_\varepsilon(s)) + u(s)]\}\,ds + \sqrt{\varepsilon}\,dW(s); \qquad y_\varepsilon(t) = x \tag{2.6}$$

where W is a H-valued Q-Wiener process in a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. A *mild* solution of (2.6) is a process $y_\varepsilon \in M^2_W(t,T;H)$ satisfying for all $s \in [t,T]$

$$y(s) = \exp[(s-t)A]\,x + \int_t^s \exp[(s-r)A]\,Q[F(r) + u(r)]\,dr + \int_t^s \exp[(s-r)A]\,dW(r)$$

where $u \in M^2_W(t,T,H)$ is the space of the H-valued processes $X(s)$ that are adapted to $W(s)$ and such that $E\int_e^T |X(t)|^2\,ds < \infty$. As easily checked by standard fixed point arguments, problem (2.6) possesses a unique mild solution.

The cost functional to be minimized is

$$J_\varepsilon(t,x,u) = \mathcal{E}\left\{\int_t^T \left[g(y_\varepsilon(s)) + \frac{1}{2}|u(s)|^2\right] ds + \phi(y_\varepsilon(T))\right\} \tag{2.7}$$

and the *Value Function* of problem (2.6), (2.7) is

$$V^\varepsilon(t,x) = \inf\{J_\varepsilon(t,x,u);\ u \in M^2_W(t,T;H)\}\,. \tag{2.8}$$

The corresponding Hamilton-Jacobi-Bellman Equation reads as follows:

$$-\frac{\partial v^\varepsilon}{\partial t} + \frac{\varepsilon}{2}\operatorname{Tr}(Qv^\varepsilon_{xx}) + \frac{1}{2}|Qv^\varepsilon_x|^2 - \langle Ax + QF(x), v^\varepsilon_x\rangle - g(x) = 0 \quad \text{in } [0,T]\times H;\ v^\varepsilon(T,x) = \phi(x)\,. \tag{2.9}$$

From Theorem 3.3 in [11] it follows that problem (2.9) has a unique *mild solution* v^ε in the space $C([0,T]\times H;\boldsymbol{R}) \cap L^\infty(0,T;C^1_Q(H,\boldsymbol{R}))$. By a mild solution of (2.9) we denote a solution of the integral equation

$$v^\varepsilon(t,\cdot) = S_{T-t}(\varphi) + \int_t^T S_{s-t}\left\{\frac{1}{2}|Qv^\varepsilon_x(s,\cdot)|^2 - \langle QF, v^\varepsilon_x(s,\cdot)\rangle - g\right\} ds \tag{2.10}$$

where S_t is the semigroup in $C_b(H,\boldsymbol{R})$ associated to the linear problem

$$\frac{\partial z}{\partial t} = \frac{\varepsilon}{2}\operatorname{Tr}(Qz_{xx}) - \langle Ax, z_x\rangle \qquad \text{in } [0,T]\times H\,. \tag{2.11}$$

For details on the construction of semigroup S_t the reader is referred to [10].

Moreover, by Proposition 4.2 in [11], the stochastic optimal control problem (2.6), (2.7) has a unique solution u^*_ε. Furthemore v^ε is equal to the value function V^ε.

3. The main result

This section is devoted to the proof of our convergence result. We start with a simple lemma.

Lemma 3.1. *Let* $u \in M^2_W(t, T; H)$, $x \in H$ *and denote by* $y(\cdot; u)$ *and* $y_\varepsilon(\cdot; u)$, *respectively, the mild solutions of the following problems*

$$(3.1) \qquad y'(s) = Ay(s) + Q[F(Y(s)) + u(s)], \qquad t \leq s \leq T; \; y(t) = x \text{ [1]}$$

$$(3.1)_\varepsilon \qquad dy_\varepsilon(s) = \{Ay_\varepsilon(s) + Q[F(y_\varepsilon(s)) + u(s)]\}\, ds + \sqrt{\varepsilon}\, dW(s), \qquad t \leq s \leq T; \; y_\varepsilon(t) = x.$$

Then,

$$(3.2) \qquad \mathcal{E}|y(s; u) - y_\varepsilon(s; u)|^2 \leq \varepsilon \operatorname{Tr}(Q) \exp[2T\|F\|_1 \|Q\|], \qquad t \leq s \leq T$$

where $\|F\|_1 = \sup_{x \in H} |F(x)| + \sup_{x \in H} |F_x(x)|$.

Proof. Let us introduce the approximating problems

$$(3.3)_n \qquad y_n'(s) = A_n y_n(s) + Q[F(y_n(s)) + u(s)], \qquad t \leq s \leq T; \; y_n(t) = x$$

$$(3.3)_{n,\varepsilon} \qquad dy_{n,\varepsilon}(s) = \{A_n y_{n,\varepsilon}(s) + Q[F(y_{n,\varepsilon}(s)) + u(s)]\}\, ds + \sqrt{\varepsilon}\, dW(s), \qquad t \leq s \leq T; \; y_{n,\varepsilon}(t) = x$$

where A_n is the Yosida approximation of A, given by $nA(n-A)^{-1}$. As easily checked by standard fixed point arguments, problems $(3.3)_n$ and $(3.3)_{n,\varepsilon}$ have unique strong solutions y_n and $y_{n,\varepsilon}$, respectively. Moreover $y_n \to y(\cdot; u)$ and $y_{n,\varepsilon} \to y_\varepsilon(\cdot; u)$ in $M^2_W(t, T; H)$, as $n \to \infty$.

Now, let $z_n = y_{n,\varepsilon} - y_n$. By the Ito formula

$$(3.4) \qquad d|z_n(s)|^2 = 2\langle z_n(s), dz_n(s)\rangle + \varepsilon \operatorname{Tr}(Q)\, ds = \{2\langle z_n(s), A_n z_n(s) + Q[F(y_{n,\varepsilon}(s)) - F(y_n(s))]\rangle + \varepsilon \operatorname{Tr}(Q)\}\, ds + 2\langle z_n(s), \sqrt{\varepsilon}\, dW(s)\rangle.$$

So

$$(3.5) \qquad \mathcal{E}|z_n(s)|^2 = \mathcal{E}\int_t^s \{2\langle z_n(r), A_n z_n(r) + Q[F(y_{n,\varepsilon}(r)) - F(y_n(r))]\rangle + \varepsilon \operatorname{Tr}(Q)\}\, ds \leq 2\|F\|_1\|Q\| \int_t^s \mathcal{E}|z_n(r)|^2\, dr + \varepsilon \operatorname{Tr}(Q).$$

By the Gronwall Lemma we have

$$(3.6) \qquad \mathcal{E}|y_n(s; u) - y_{n,\varepsilon}(s; u)|^2 \leq \varepsilon \operatorname{Tr}(Q) \exp[2T\|F\|_1\|Q\|], \qquad t \leq s \leq T$$

and the conclusion follows letting n tend to infinity. ■

We prove now the main result of the paper:

[1] Since u is a stochastic process, (3.1) has to be solved for any $\omega \in \Omega$.

Theorem 3.2. Let $V(t,x)$ be given by (2.4) and v^{ε} be the mild solution of problem (2.9). Them, $\lim_{\varepsilon \downarrow 0} v^{\varepsilon}(t,x) = V(t,x)$ uniformly for $(t,x) \in [0,T] \times H$.

Proof. Recalling the remarks at the end of section 2, it suffices to show that $\lim_{\varepsilon \downarrow 0} V^{\varepsilon}(t,x) = V(t,x)$, uniformly for $(t,x) \in [0,T] \times H$, where $V^{\varepsilon}(t,x)$ is given by (2.8). For this purpose, we follow the argument of [12]. Let $\{u^*, y^*\}$ (resp. $\{u_{\varepsilon}^*, y_{\varepsilon}^*\}$) be an optimal pair for V (resp. V^{ε}) at (t,x). We have:

$$8J(t,x,u_{\varepsilon}^*) \geq V(t,x); \qquad V^{\varepsilon}(t,x) \leq J_{\varepsilon}(t,x,u^*) \tag{3.7}$$

then

$$J_{\varepsilon}(t,x,u_{\varepsilon}^*) - 8J(t,x,u_{\varepsilon}^*) \leq V^{\varepsilon}(t,x) - V(t) \leq J_{\varepsilon}(t,x,u^*) - J(t,x,u^*)\,. \tag{3.8}$$

By Lemma 3.1 we obtain:

$$J_{\varepsilon}(t,x,u_{\varepsilon}^*) - 8J(t,x,u_{\varepsilon}^*) = 8\left\{\int_t^T [g(y_{\varepsilon}(s,u_{\varepsilon}^*)) - g(y(s,u_{\varepsilon}^*))]\,ds + {} + \phi(y_{\varepsilon}(T,u_{\varepsilon}^*)) - \phi(y(T,u_{\varepsilon}^*))\right\} \to 0 \qquad \text{as } \varepsilon \to 0, \tag{3.9}$$

$$J_{\varepsilon}(t,x,u^*) - J(t,x,u^*) = 8\left\{\int_t^T [g(y_{\varepsilon}(s,u^*)) - g(y(s,u^*))]\,ds + {} + \phi(y_{\varepsilon}(T,u^*)) - \phi(y(T,u^*))\right\} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.10}$$

The conclusion follows from (3.8), (3.9) and (3.10). ■

References

[1] Barbu V. and Da Prato G., 1983. *Solution of the Bellman equation associated with an infinite dimensional stochastic control problem and synthesis of optimal control.* SIAM J. Control Opt., 21, 4: 531-550.

[2] Cannarsa P. and Da Prato G. *Some results on nonlinear optimal control problems and Hamilton-Jacobi equations in infinite dimensions.* J. Funct. Anal. (to appear).

[3] Cannarsa P. and Da Prato G., 1989. *Nonlinear optimal control with infinite horizon for distributed parameter systems and stationary Hamilton-Jacobi equations.* SIAM J. Control Opt., 27, 4: 861-875.

[4] Crandall M. G. and Lions P. L., 1983. *Viscosity solutions of Hamilton-Jacobi equations.* Trans. Amer. Math. Soc., 277: 183-186.

[5] Crandall M. G. and Lions P. L., 1985. *Hamilton-Jacobi equations in infinite dimensions Part I. Uniqueness of Viscosity Solutions.* J. Funct. Anal., 62: 379-396.

[6] Crandall M. G. and Lions P. L., 1986. *Hamilton-Jacobi equations in infinite dimensions. Part II. Existence of Viscosity Solutions.* J. Funct. Anal., 65: 368-405.

[7] Crandall M. G. and Lions P. L., 1986. *Hamilton-Jacobi equations in infinte dimensions. Part III.* J. Funct. Anal., 68: 368-405.

[8] Crandall M. G. and Lions P. L., 1987. *Solutions de visconsité pour les équations de Hamilton-Jacobi en dimension infinie intervenant dans le contrôle optimal des problèmes d'evolution.* C.R. Acad. Sci. Paris, 305: 233-236.

[9] Daleckii J. L., 1966. *Differential equations with functional derivatives and stochastic equations for generalized random processes.* Dokl. Akad. Nauk SSSR, 166: 1035-1038.

[10] Da Prato G., 1987. *Some Results on Parabolic Evolution Equations with Infinitely Many Variables.* J. Differential Equations, 68, 2: 281-297.

[11] Da Prato G., 1985. *Some results on Bellman equation in Hilbert spaces and applications to infinite dimensional control problems.* In «Stochastic Differential Systems, Filtering and Control», Lecture Notes in Control and Information Sciences n° 69, Proceedings of the IFIP-WG 7/1 Working Conference, Marseille-Luminy, France, March 12-17, 1984. Mativier M. and Pardoux E. Editors: 270-280.

[12] Fleming W. H., 1969. *The Cauchy problem for a nonlinear first order partial differential equation.* J. Differential Equations, 5: 515-530.

[13] Gross L., 1967. *Potential theory in Hilbert space.* J. Func. Anal., 1: 123-181.

[14] Havarneanu T., 1985. *Existence for the Dynamic Programming equations of control diffusion processes in Hilbert spaces.* Nonlinear Anal. T.M.A, 9, n° 6: 619-629.

[15] Lions P. L., 1982. *Generalized solutions of Hamilton-Jacobi equations.* Pitman, Boston.

[16] Piech M. A., 1969. *A Fundamental Solution of the Parabolic Equations in Hilbert spaces.* J. Funct. Analysis, 3: 85-114.