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Evolution equations in non-cylindrical domains

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Analisi matematica. — *Evolution equations in non-cylindrical domains.* Nota di
Piermarco Cannarsa, Giuseppe Da Prato e Jean-Paul Zolésio, presentata (*) dal Corrisp. A. Ambrosetti.

Abstract. — We develolp a new method to solve an evolution equation in a non-cylindrical domain, by reduction to an abstract evolution equation..

Key words: Evolution equations; Maing domains; Damped wave equation.

Riassunto. — *Equazioni di evoluzione in domini non cilindrici.* Si dà un nuovo metodo per risolvere un'equazione di evoluzione in un dominio non cilindrico, riconducendola a un'equazione astratta.

1. Heat equation in non-cylindrical domains

We are here concerned with the following problem:

$$\begin{cases} y_t(t,x) = \Delta y(t,x) + v(t,x) & \text{in } Q_T, \\ y(0,x) = y_0(x) & \text{in } \Omega_0, \\ y(t,x) = 0 & \text{in } \Sigma_T, \end{cases} \tag{1.1}$$

where Δ denotes the Laplace operator on a smooth domain Ω_t in $\boldsymbol{R}^N$ and Q_T is the non-cylindrical evolution domain in $\boldsymbol{R}^{N+1}$

$$Q_T = \bigcup_{0<t<T} \{t\} \times \Omega_t . \tag{1.2}$$

Moreover the lateral boundary Σ_T is defined as:

$$\Sigma_T = \bigcup_{0<t<T} \{t\} \times \Gamma_t \tag{1.3}$$

where Γ_t is the boundary of Ω_t.

Our approach consists of reducing problem (1.1) to an abstract evolution equation:

$$y'(t,\cdot) = A(t)\, y(t,\cdot) + v(t,\cdot), \qquad y(0,\cdot) = y_0 \tag{1.4}$$

in the space $X = L^2(\boldsymbol{R}^N)$, with norm:

$$\|y\| = \left[\int_{\boldsymbol{R}^N} |y(x)|^2 \, dx \right]^{1/2} .$$

The linear operators $A(t)$ are defined as follows:

$$D(A(t)) = \{y \in X; y_{|\Omega_t} \in H^2(\Omega_t) \cap H_0^1(\Omega_t), \, y_{|\Omega_t^c} \in H^2(\Omega_t^c) \cap H_0^1(\Omega_t^c)\} \tag{1.5}$$

where Ω_t^c is the complementary set in $\boldsymbol{R}^N$ of Ω_t. Moreover

$$\int_{\boldsymbol{R}^N} A(t) z \phi \, dx = \int_{\boldsymbol{R}^N} z \Delta \phi \, dx , \qquad \text{for all } \phi \in \mathcal{D}(\boldsymbol{R}^N) \text{ such that } \phi = 0 \text{ on } \Gamma_t . \tag{1.6}$$

(*) Nella seduta del 26 novembre 1988.

LEMMA 1. For all $t \geq 0$, $A(t)$ is the infinitesimal generator of an analytic semigroup in X.

PROOF. We have to show that the problem:

$$\lambda u(t,\cdot) - A(t)\,u(t,\cdot) = f(\cdot)\,, \qquad f \in X \tag{1.7}$$

has a unique solution $u(t,\cdot) \in D(A(t))$ for all λ in the sector $S = \{\lambda \in \mathbb{C};\, |\arg\lambda| < \pi\}$ and that the resolvent estimate holds, i.e.

$$\|(\lambda - A(t))^{-1}\| \leq M/|\lambda|\,. \tag{1.8}$$

In fact, problem (1.7) is equivalent to:

$$\begin{cases} (a)\ \lambda u_1(t,x) - \Delta u_1(t,x) = f(x), & u_1(t,x) \in H^2(\Omega_t), \quad u_1(t,\cdot) = 0 \quad \text{on } \Gamma_t\,, \\ (b)\ \lambda u_2(t,x) - \Delta u_2(t,x) = f(x), & u_2(t,x) \in H^2(\Omega_t^c), \quad u_2(t,\cdot) = 0 \quad \text{on } \Gamma_t\,. \end{cases} \tag{1.9}$$

Now, we apply the results of Agmon [3] to each of the problems in (1.9) and obtain the conclusion. #

Next, we not that the domains $D(A(t))$ are not constant. So, we are naturally led to use the Kato-Tanabe approach [4] in the improved form of Acquistapace-Terreni [1]. For this purpose we have to obtain the following estimate:

$$\left\| \frac{d}{dt}(\lambda - A(t)^{-1}) \right\| \leq \frac{M}{|\lambda - \omega|^\beta} \tag{1.10}$$

for some $\omega \in \boldsymbol{R}$, $\beta > 0$ and all λ satisfying $\operatorname{Re}\lambda > \omega$. Therefore, we will majorize the norms of u_1, u_2 and of their derivatives with respect to the parameter t.

Following [5], we assume throughout that Ω_t can be constructed by using a smooth transformation T_t of $\boldsymbol{R}^N$ into itself as follows. We assume given a continuously differentiable function $V(t,x)$ defined on $[0,T] \times \boldsymbol{R}^N$ such that $\nabla V(t,\cdot)$ is Lipschitz and consider the associated flow which we call T_t. Therefore we have:

$$V(t,x) = \left(\frac{\partial}{\partial t} T_t\right) \circ T_t^{-1}(x)\,. \tag{1.11}$$

Now, we assume:

$$\Omega_t = T_t(\Omega_0)\,. \tag{1.12}$$

Let us introduce some additional notation. We set:

$$J_t = \det(DT_t) \tag{1.13}$$

$$\Lambda(t) = J_t^*(DT_t)^{-1}(DT_t)^{-1} \tag{1.14}$$

where $^*(DT_t)^{-1}$ denote the transposed matrix of $(DT_t)^{-1}$.

Now, consider the resolvent equation (1.7) and split it in the form (1.9). In the sequel, we denote by O_t either one of the set Ω_t and Ω_t^c. One can show that there exists the derivative $u_t(t,x)$ (with respect to t) and

$$u_t(t,x) = \dot{u}(t,x) - \nabla u(t,x) \cdot V(t,x) \tag{1.15}$$

where · denotes the scalar product in R^N and $\dot{u}$ is the solution of the problem

(1.16) $$\lambda\dot{u} - \Delta\dot{u} = \operatorname{div}(fV(t)) - u\operatorname{div}(V(t)) + \operatorname{div}(\Lambda'(t)\nabla u), \qquad \text{in } O_t.$$

We can now prove (1.10).

PROPOSITION 2. There exist $\omega > 0$ such that

(1.17) $$\left\|\frac{d}{dt}(\lambda - A(t)^{-1})\right\| \le \frac{M}{\sqrt{|\lambda - \omega|}}, \qquad \text{for all } \operatorname{Re}\lambda > \omega \text{ and a suitable constant } M.$$

PROOF. First recall that by the results of S. Agmon [3[, the following estimates hold for the solutions u_1, u_2 of problems (1.9)-*a*)-*b*).

(1.18) $$|\lambda|\,\|u_1(t,\cdot)\|_{L^2(\Omega_t)} + \sqrt{|\lambda|}\,\|\nabla u_1(t,\cdot)\|_{L^2(\Omega_t)} \le C\|f\|_{L^2(\Omega_t)}$$

(1.19) $$|\lambda|\,\|u_2(t,\cdot)\|_{L^2(\Omega_t^c)} + \sqrt{|\lambda|}\,\|\nabla u_2(t,\cdot)\|_{L^2(\Omega_t^c)} \le C\|f\|_{L^2(\Omega_t^c)}$$

where C is a suitable constant $\operatorname{Re}\lambda > 0$.

By (1.16) we have:

(1.20) $$\lambda\int_{O_t}\dot{u}(t,x)\,\overline{\phi(x)}\,dx + \int_{O_t}\nabla\dot{u}(t,x)\cdot\nabla\overline{\phi(x)}\,dx =$$
$$= -\int_{O_t} f(x)\,V(t,x)\cdot\nabla\overline{\phi(x)}\,dx - \int_{O_t} u(t,x)\operatorname{div}V(t,x)\,\overline{\phi(x)}\,dx - \int_{O_t}\Lambda'(t)\,\nabla u(t,x)\cdot\nabla\overline{\phi(x)}\,dx$$

for all $\phi \in H_0^1(O_t)$, whee $O_t = \Omega_t$ or Ω_t^c.

By taking $\phi = \dot{u}$ in (1.20) with $O_t = \Omega_t$ or Ω_t^c, and adding up, we find:

(1.21) $$\lambda\int_{R^N}|\dot{u}(t,x)|^2dx + \int_{R^N}|\nabla\dot{u}(t,x)|^2\,dx = -\int_{R^N} f(x)\,V(t,x)\,\overline{\nabla\dot{u}(t,x)}\,dx -$$
$$-\int_{R^N}\Lambda'(t)\,\nabla u(t,x)\cdot\overline{\nabla\dot{u}(t,x)}\,dx - \int_{R^N} u(t,x)\operatorname{div}V(t,x)\,\overline{\dot{u}(t,x)}\,dx.$$

Now, taking the real part of (1.21) and recalling (1.18) and (1.19), we have, by standard computations, that there exist $\omega > 0$ such that, if $\operatorname{Re}\lambda > \omega$, then

(1.22) $$\int_{R^N}|\nabla\dot{u}(t,x)|^2\,dx \le C\int_{R^N}|f(x)|^2\,dx$$

where C is a suitable constant.

Finally, going back to (1.21) and passing to absolute values, we obtain the estimate:

(1.23) $$\|\dot{u}(t,\cdot) \le C\frac{\|f\|}{\sqrt{\lambda - \omega|}}.$$

The conclusion follows by (1.15), (1.18), (1.19) and (1.23). #

Now, by the abstract results of [1] and [2], the theorem below follows.

THEOREM 3. For any $y_0 \in X$ and $v \in C([0,T];X)$ problem (1.4) has a unique strong solution y. Moreover, for all $\alpha \in]0,1[$, we have $y \in C^\alpha([0,T];X)$ and $y(t) \in D_{A(t)}(\alpha, 2)$ (the Lions-Peetre interpolation spaces).

REMARK 4. From Theorem 3 we conclude that the function $y(t,x) = y(t)(x)$ solves problem (1.1) in the following sense:

$$\int_{Q_T} \nabla y \cdot \nabla \phi \, dx \, dt - \int_{Q_T} y \phi_t \, dx \, dt = \int_{Q_T} v \phi \, dx \, dt + \int_{Q_T} y_0 \phi(0,\cdot) \, dx \tag{1.24}$$

for all $\phi \in C^\infty(Q_T)$ such that $\phi(t,x) = 0$ on $(\{T\} \times \Omega_T) \cup \Sigma_T$. In a forthcoming paper we shall treat more general elliptic operators and give a more detailed analysis on the regularity properties of y. #

2. The damped wave equation in non-cylindrical domains

We consider the following problem:

$$\begin{cases} y_{tt}(t,x) = \Delta y(t,x) + \Delta y_t(t,x) + v(t,x) & \text{in } Q_T, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x) & \text{in } \Omega_0, \\ y(t,x) = 0, & \text{in } \Sigma_T, \end{cases} \tag{2.1}$$

where we keep the notation of the previouos section.

By writing:

$$Y(t) = \begin{bmatrix} y(t,\cdot) \\ y_t(t,x) \end{bmatrix} \tag{2.2}$$

problem (2.1) can be set, in the Banach space $Z = X \oplus X$, in the following abstract form:

$$Y'(t) = \mathcal{A}(t)\, Y(t) + V(t), \qquad 0 \le t \le T;\; Y(0) = Y_0 \tag{2.3}$$

where:

$$\mathcal{A}(t) = \begin{bmatrix} 0 & 1 \\ A(t) & A(t) \end{bmatrix}; \quad V(t) = \begin{bmatrix} 0 \\ v(t,\cdot) \end{bmatrix}; \quad Y_0 = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \tag{2.4}$$

As easily checked, the resolvent set $\rho(\mathcal{A}(t))$ of $\mathcal{A}(t)$ contains a sector $S_\theta = \{\lambda \in wC; |\arg \lambda| < \theta\}$, with some $\theta \in]\pi/2, \pi[$. Moreover, we have:

$$(\lambda - \mathcal{A}(t))^{-1} = \begin{bmatrix} R_{11}(t,\lambda) & R_{12}(t,\lambda) \\ R_{21}(t,\lambda) & R_{22}(t,\lambda) \end{bmatrix}; \qquad \forall t \ge 0,\ \forall \lambda \in S_\theta \tag{2.5}$$

where:

$$R_{11}(t,\lambda) = (\lambda - A(t))(\lambda^2 - \lambda A(t) - A(t))^{-1} = \frac{1}{\lambda+1} + \frac{\lambda}{(\lambda+1)^2}\left(\frac{\lambda^2}{\lambda+1} - A(t)\right)^{-1} \tag{2.6}$$

$$R_{12}(t,\lambda) = (\lambda^2 - \lambda A(t) - A(t))^{-1} = \frac{1}{\lambda+1}\left(\frac{\lambda^2}{\lambda+1} - A(t)\right)^{-1} \tag{2.7}$$

$$R_{21}(t,\lambda) = A(t)(\lambda^2 - \lambda A(t) - A(t))^{-1} = -\frac{1}{\lambda+1} + \frac{\lambda^2}{(\lambda+1)^2}\left(\frac{\lambda^2}{\lambda+1} - A(t)\right)^{-1} \tag{2.8}$$

$$R_{22}(t,\lambda) = \lambda(\lambda^2 - \lambda A(t) - A(t))^{-1} = \frac{\lambda}{\lambda+1}\left(\frac{\lambda^2}{\lambda+1} - A(t)\right)^{-1}. \tag{2.9}$$

LEMMA 5. $\mathcal{A}(t)$ generates an analytic semigroup in Z and the following estimates hold:

$$\|(\lambda - \mathcal{A}(t))^{-1}\| \le \frac{M_1}{|\lambda - \omega_1|}; \quad \left\| \frac{d}{dt}(\lambda - \mathcal{A}(t))^{-1} \right\| \le \frac{M_1}{\sqrt{|\lambda - \omega_1|}}; \quad \mathrm{Re}\,\lambda > \omega_1 \tag{2.10}$$

where M_1 and ω_1 are suitable constants.

PROOF. By using (1.8) and (1.17) to estimate the norms of R_{ij} and their derivatives with respect to t we obtain:

$$\|R_{ij}(t,\lambda)\| \le \frac{C}{|\lambda|}; \quad \left\| \frac{\partial}{\partial t} R_{ij}(t,\lambda) \right\| \le \frac{C}{\sqrt{|\lambda - \omega_1|}}; \quad i,j = 1,2, \ \mathrm{Re}\,\lambda > \omega_1 \tag{2.11}$$

for some positive constant C and ω_1. Now the conclusion (2.10) follows from (2.5) and (2.11). #

The following existence and uniqueness result can be deduced by Lemma 5 and the abstract theory of [1], [2].

THEOREM 6. Let $y_0, y_1 \in X$, $v \in C([0,T];X)$. Then problem (2.3) has a unique srong solution Y. Moreover, for all $\alpha \in]0,1[$, ϑε ηαωε $Y \in C^\alpha([0,T];Z)$ and $Y(t) \in D_{\mathcal{A}(t)}(\alpha, 2)$.

REMARK 7. Theorem 6 implies that for any $y_0, y_1 \in X$, $v \in C([0,T];X)$, problem (2.1) has a unique solution in the distribution sense. In a forthcoming paper we shall analyze this example and derive further regularity of the solution. #

REMARK 8. Notice that problem (2.1) seems hard approach by change of variable techniques. In fact, such a technique uses a transformation that, by adding a third order term in the x-derivatives, destroys the parabolic nature of the equation. #

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