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Analisi matematica. — *On some identities involving spherical means.* Nota (*) del Socio GIANFRANCO CIMMINO.

ABSTRACT. — For every positive definite quadratic form in n variables the reciprocal of the square root of the discriminant is equal to the arithmetic mean of the values assumed by the form on the $n-1$ sphere centered at O and with radius 1 raised to the $(-n/2)$-th power. Various consequences are deduced from this, in particular a simplification of some calculations from which one obtains the possibility of solving linear systems using spherical means rather than determinants.

KEY WORDS: Spherical means; Quadratic forms; Linear systems.

RIASSUNTO. — *A proposito di alcune identità relative a medie sferiche.* Per ogni forma quadratica definita positiva in n variabili il discriminante elevato a $-1/2$ uguaglia la media aritmetica dei valori assunti sulla varietà sferica $(n-1)$-dimensionale di centro O e raggio 1 elevati a $-n/2$. Da ciò si deducono varie conseguenze, in particolare una semplificazione dei calcoli da cui risulta la possibilità di risolvere i sistemi lineari usando medie sferiche, anziché determinanti.

1. Let us call ω_n the set of all vectors $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ for which

$$\|\xi\|^2 = \xi_1^2 + \dots + \xi_n^2 = 1 . \tag{1}$$

For any continuous function $f(\xi)$, $\xi \in \omega_n$ we define $\mathfrak{M} f(\xi)$ by

$$\mathfrak{M} f(\xi) = \frac{1}{|\omega_n|} \int_{\omega_n} f(\xi)\, d\omega_n, \qquad |\omega_n| = \int_{\omega_n} d\omega_n = \frac{(\sqrt{2\pi})^{n-1}}{(n-2)!!} \begin{cases} \sqrt{2\pi}, & n \text{ even}, \\ 2, & n \text{ odd}. \end{cases} \tag{2}$$

This $\mathfrak{M} f$ is what we call the *spherical mean* of $f(\xi)$, thinking of the case $n > 2$. We are here concerned with extensions to this case of some noteworthy formulas valid for $n = 2$.

For example, if $a, b, c,$ are real numbers with $a > 0$, $ac - b^2 > 0$, the equality

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{a\,\mathrm{c}^2\varphi + 2b\mathrm{c}\varphi\,\mathrm{s}\varphi + c\,\mathrm{s}^2\varphi} = \frac{1}{\sqrt{ac - b^2}} \tag{3}$$

can be generalized to the case of any quadratic form

$$\sum_{hk=1}^{n} a_{hk}\xi_h\xi_k = \langle A\xi, \xi\rangle , \qquad n \geq 2 , \tag{4}$$

with A symmetric $n \times n$ real matrix such that

$$\min_{\xi \in \omega_n} \langle A\xi, \xi\rangle > 0 , \tag{5}$$

by writing

$$\mathfrak{M}(\langle A\xi, \xi\rangle^{-n/2}) = (\det A)^{-1/2} . \tag{6}$$

We are at first going to show that this is a consequence of what I noted in my articles [1], [2], and conversely.

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2. As may be seen in [2], for any $n \times n$ real matrix A' with $\det A' \neq 0$, one has the following equivalence (where $\overline{A}'$ is the transpose of A')

$$\overline{A}'x = b \Leftrightarrow x = n\frac{\mathcal{M}(\|A'\xi\|^{-n-2}\langle b,\xi\rangle A'\xi)}{\mathcal{M}(\|A'\xi\|^{-n})}, \tag{7}$$

and $|\det A'|$ can be obtained from the identity

$$\mathcal{M}(\|A'\xi\|^{-n}) = |\det A'|^{-1}. \tag{8}$$

Now, as is well know, each symmetric A verifying (5) has a unique positive square root A', for which

$$\overline{A}' = A', \qquad A'^2 = A, \qquad (\det A')^2 = \det A, \qquad \|A'\xi\|^2 = \langle A\xi,\xi\rangle. \tag{9}$$

Therefore the validity of (6) for every such symmetric A appears as a consequence of (8).

Conversely, this validity implies the equality (8) to be verified, whenever $\det A' \neq 0$, since $A'\overline{A}'$ is then positive symmetric, and one gets (8) by putting $A = A'\overline{A}'$, $\langle A\xi,\xi\rangle = \|A'\xi\|^2$, $\det A = (\det A')^2$ in (6).

3. The equality (3) happened to be a useful tool in another paper of mine [3], giving rise to a sequence of equalities occurring there. We can get the corresponding sequence in the case $n \geq 3$, by iterating on both sides of (6) a linear differential operator of the type

$$\sum_{hk=1}^{n} b_{hk}\frac{\partial}{\partial a_{hk}} \tag{10}$$

where the coefficients b_{hk}, $h, k = 1, \ldots, n$ form any symmetric matrix B. This yields at first

$$n\,\mathcal{M}(\langle A\xi,\xi\rangle^{-n/2-1}\langle B\xi,\xi\rangle) = (\det A)^{-3/2}\sum_{hk=1}^{n} A_{hk}b_{hk}, \tag{11}$$

where A_{hk} indicates the cofactor of a_{hk} in A.

The equalities one then obtains by iteration of (10) reduced to very simple ones in the particular case that, for some pair of vectors $b = (b_1, \ldots, b_n)$, $b' = (b'_1, \ldots, b'_n)$,

$$b_{hk} = b_h b'_k, \qquad h, k = 1, \ldots, n. \tag{12}$$

In consequence of (12) the equality (11) becomes

$$n\,\mathcal{M}(\langle A\xi,\xi\rangle^{-n/2-1}\langle b,\xi\rangle\langle b',\xi\rangle) = \langle A^{-1}b, b'\rangle(\det A)^{-1/2} \tag{13}$$

and furthermore, if one applies m times the differential operator (10), one gets

$$\frac{n(n+2)\ldots(n+2m-2)}{1\cdot 3\cdot\ldots\cdot(2m-1)}\mathcal{M}(\langle A\xi,\xi\rangle^{-n/2-m}\langle b,\xi\rangle^m\langle b',\xi\rangle^m) = \tag{14}$$

$$= \langle A^{-1}b, b'\rangle^m(\det A)^{-1/2},$$

because of the fact that when b_{hk} has the particular form (12), then by applying operator (10) to $\sum_{hk=1}^{n} A_{hk}b_h b'_k$ one finds 0.

The equality (13) for every symmetric positive A is a consequence of (7), whence it immediately follows if one takes A' satisfying (9) and one then considers the inner products by $A'^{-1}b'$ on both sides.

Conversely, if (13) holds for every symmetric positive A, then (7) will hold for every A' with $\det A' \neq 0$, as follows by putting $A = A'\overline{A}'$ in (13) and by choosing b' to give successively the n rows of A'.

4. The identity (14) for every $A = \overline{A}$ verifying (5) can also be proved as follows. Replace a_{hk} by $a_{hk} - tb_h b'_k$ in (6), with t small enough in order to ensure that the power series

$$(1 - z)^{-n/2} = 1 + \frac{n}{2}z + \ldots + \frac{n(n+2)\ldots(n+2m-2)}{2\cdot 4\cdot\ldots\cdot 2m}z^m + \ldots \tag{15}$$

converges for $z = \langle b, \xi\rangle\langle b', \xi\rangle t/\langle A\xi, \xi\rangle$ and for $z = \langle A^{-1}b, b'\rangle \cdot t$, and in the resulting equality

$$\mathcal{M}[(\langle A\xi, \xi\rangle - t\langle b, \xi\rangle\langle b', \xi\rangle)^{-n/2}] = (\det A)^{-1/2}(1 - \langle A^{-1}b, b'\rangle\, t)^{-1/2} \tag{16}$$

take on both sides the Taylor expansions with respect to t. The two power series one gets by this ought to have the same coefficients, thus (14) must be verified $\forall m \in \mathbb{N}^+$.

More generally, let

$$\varphi(z) = 1 + c_1 z + c_2 z^2 + \ldots \tag{17}$$

be any power series with $\varphi(0) = 1$, having a positive convergence radius, and let us define $\varphi_n(z)$ by

$$\varphi_n(z) = 1 + \sum_{m=1}^{\infty} \frac{n(n+2)\ldots(n+2m-2)}{1\cdot 3\cdot\ldots\cdot(2m-1)} c_m z^m, \qquad \forall n \in \mathbb{N}^+ \tag{18}$$

so that $\varphi_1(z) = \varphi(z)$. If now we multiply both sides of (14) by c_m, we shall get, by addition with respect to m, the identity

$$\mathcal{M}\left[\langle A\xi, \xi\rangle^{-n/2}\varphi_n\left(\frac{\langle b, \xi\rangle\langle b', \xi\rangle}{\langle A\xi, \xi\rangle}\right)\right] = \varphi(\langle A^{-1}b, b'\rangle)(\det A)^{-1/2}, \tag{19}$$

which reduces to (16) in the particular case $\varphi(z) = (1 - tz)^{-1/2}$.

5. Further identities of this type can be deduced from (6). For example, let us replace a_{hk} by $\delta_{hk} - \lambda a_{hk}$ for any $\lambda \in \mathbb{C}$ such that $\det(I - \lambda A) \neq 0$. We get

$$\mathcal{M}(1 - \lambda\langle A\xi, \xi\rangle)^{-n/2} = [\det(I - \lambda A)]^{-1/2}. \tag{20}$$

And for $|\lambda|$ sufficiently small we can take on both sides the Taylor expansions. By remembering (15) and the usual equality

$$\det(I - \lambda A) = 1 - \alpha_1\lambda + \alpha_2\lambda^2 - \ldots + (-1)^n\alpha_n\lambda^n, \tag{21}$$

where $\alpha_n = \det A$ and $\alpha_1, \ldots, \alpha_{n-1}$ are the so-called orthogonal invariants of the matrix A, we shall obtain the following identity

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{n(n+2)\ldots(n+2m-2)}{2\cdot 4\cdot\ldots\cdot(2m)} \mathcal{M}(\langle A\xi, \xi\rangle^m)\lambda^m &= \\ = \sum_{m=1}^{\infty} \frac{(2m-1)!!}{(2m)!!}(\alpha_1\lambda - \alpha_2\lambda^2 + \ldots &+ (-1)^{n-1}\alpha_n\lambda^n)^m, \end{aligned} \tag{22}$$

where, by reordering the right side into a power series of λ, we must find the same coefficients as those of the power series on the left side.

6. In (22) we meet spherical means of polynomials. We end with some remarks concerning them.

Firstly, the spherical mean of $\xi_1^{m_1} \dots \xi_n^{m_n}$ with $m_1, \dots, m_n$ integers ≥ 0 is $\neq 0$ only when $m_1, \dots, m_n$ are all even. Secondly, if $m_1', \dots, m_n'$ is any permutation of $m_1, \dots, m_n$, then $\mathcal{M}(\xi_1^{m_1} \dots \xi_n^{m_n}) = \mathcal{M}(\xi_1^{m_1'} \dots \xi_n^{m_n'})$. This is a consequence of the fact that if one parametric representation of ω_n is given, then other ones can be obtained from it by taking $-\xi_k$ instead of ξ_k, for any $k = 1, \dots, n$, or by permutating $\xi_1, \dots, \xi_n$ as one likes.

Further, by calculating, one finds that

$$\mathcal{M}(\xi_1^{2k_1} \dots \xi_n^{2k_n}) = \frac{(2k_1 - 1)!! \dots (2k_n - 1)!! \, (n-2)!!}{(2k_1 + \dots + 2k_n + n - 2)!!}, \qquad \forall (k_1, \dots, k_n) \in \mathbb{N}^n. \tag{23}$$

Therefore, because of (1) and (2),

$$\sum_{k_1 + \dots + k_n = m} \frac{m! \, (2k_1 - 1)!! \dots (2k_n - 1)!!}{k_1! \dots k_n!} = \frac{(2m + n - 2)!!}{(n-2)!!}, \tag{24}$$

and also, since $e^{\xi_1^2 z} \dots e^{\xi_n^2 z} = e^z$,

$$\sum_{k_1, \dots, k_n = 0}^{\infty} \frac{(2k_1 - 1)!! \dots (2k_n - 1)!! \, z^{k_1 + \dots k_n}}{(2k_1 + \dots + 2k_n + n - 2)!! \, k_1! \dots k_n!} = \frac{e^z}{(n-2)!!}. \tag{25}$$

And one can note that the identity (25) derives from (24) as a consequence of the fact that $\sum_{k_1, \dots, k_n = 0}^{\infty} = \sum_{m=0}^{\infty} \sum_{k_1 + \dots + k_n = m}$.

In general, for every absolutely convergent multiple power series of $\xi_1^2, \dots, \xi_n^2$, one gets from (24) the value of the corresponding spherical mean in the form of another multiple series.

It is to be remarked that, for every $f(x)$, harmonic function of $x = (x_1, \dots, x_n)$ in an open set of $\mathbb{R}^n$ containing the ball $\|x\| \leq 1$, the spherical mean as defined by (1) must be $= f(0)$. In particular, the spherical mean of each homogeneous harmonic polynomial not reducing to a constant will be $= 0$.

References

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[3] Cimmino G., 1988. *A conjecture on minimal surfaces.* Rend.Acc. Naz. Lincei, s. 8, LXXXII, in corso di stampa.