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Two theorems on the Scorza Dragoni property for multifunctions

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Analisi matematica. — *Two theorems on the Scorza Dragoni property for multifunctions.* Nota (*) di GABRIELE BONANNO, presentata dal Socio G. SCORZA DRAGONI.

ABSTRACT. — We point out two theorems on the Scorza Dragoni property for multifunctions. As an application, in particular, we improve a Carathéodory selection theorem by A. Cellina [4], by removing a compactness assumption.

KEY WORDS: Multifunction; Scorza Dragoni property; Carathéodory selection.

RIASSUNTO. — *Due teoremi sulla proprietà di Scorza Dragoni per le multifunzioni.* Segnalo due teoremi sulla proprietà di Scorza Dragoni per le multifunzioni. Come applicazione, in particolare, miglioro un teorema di A. Cellina [4] sulle selezioni di Carathéodory, rimuovendo un'ipotesi di compattezza.

Here and in the sequel, T is a Hausdorff topological space; (X, d), (Y, ρ) are two metric spaces, with X separable; $\mathcal{F}$ is a σ-algebra of subsets of T containing the Borel family of T; μ is a finite measure on $\mathcal{F}$ such that $\mu(A) = \sup\{\mu(K): K \subseteq A, K \text{ compact}\}$ for every $A \in \mathcal{F}$.

The aim of this paper is to point out the two following theorems.

THEOREM 1. *Let F be a multifunction from $T \times X$ into Y satisfying the following conditions*:

(*a*) *for every $t \in T$, the multifunction $F(t, \cdot)$ is continuous;*

(*b*) *the set $\{x \in X$: the multifunction $F(\cdot, x)$ is $\mathcal{F}$-measurable and the set $F(T, x)$ is separable$\}$ is dense in X.*

Then, for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq T$, with $\mu(T \setminus K_\varepsilon) < \varepsilon$, such that the multifunction $F|_{K_\varepsilon \times X}$ is lower semicontinuous and the set $\{(t, x, y) \in K_\varepsilon \times X \times Y: y \in \overline{F(t, x)}\}$ is closed in $K_\varepsilon \times X \times Y$.

THEOREM 2. *Let F be a multifunction from $T \times X$ into Y satisfying the following conditions*:

(*a*) *for every $t \in T$, the multifunction $F(t, \cdot)$ is lower semicontinuous;*

(*b*) *for every $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{F}$, with $\mu(T \setminus A_\varepsilon) < \varepsilon$, such that, for each $x \in X$, the multifunction $F(\cdot, x)|_{A_\varepsilon}$ is upper semicontinuous and the set $F(A_\varepsilon, x)$ is separable.*

Then, for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq T$, with $\mu(T \setminus K_\varepsilon) < \varepsilon$, such that the multifunction $F|_{K_\varepsilon \times X}$ is lower semicontinuous.

The paper is arranged as follows. Section 1 contains definitions and preliminary propositions. The proofs of Theorems 1 and 2 are given in Section 2. Finally, some remarks and applications are put in Section 3. In particular, we improve a selection theorem of A. Cellina [4], by removing a compactness assumption.

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1. Preliminaries

Let S, V be two topological spaces and let Φ be a multifunction from S into V, that is a function from S into the family of all non-empty subsets of V. For each $A \subseteq S$, $\Omega \subseteq V$, we put $\Phi(A) = \bigcup_{x \in A} \Phi(x)$ and $\Phi^-(\Omega) = \{x \in S: \Phi(x) \cap \Omega \neq \emptyset\}$. We say that Φ is lower (resp. upper) semicontinuous if, for every open (resp. closed) set $\Omega \subseteq V$, the set $\Phi^-(\Omega)$ is open (resp. closed) in S. We say that Φ is continuous if it is simultaneously lower and upper semicontinuous. The notions of lower and upper semicontinuity for real (single-valued) functions are the usual ones. If $\mathcal{G}$ is a σ-algebra of subsets of S, we say that Φ is $\mathcal{G}$-measurable if $\Phi^-(\Omega) \in \mathcal{G}$ for every open set $\Omega \subseteq V$. For a set A in a topological space, $\overline{A}$ denotes its closure. If (Σ, δ) is a metric space, given $x \in \Sigma$, $r > 0$, $A \subseteq \Sigma$ non-empty, we put: $B_\delta(x, r) = \{y \in \Sigma: \delta(x, y) < r\}$ and $\delta(x, A) = \inf_{y \in A} \delta(x, y)$.

The next propositions will be used in the proofs of Theorems 1 and 2. Let us recall that the meaning of T, $\mathcal{F}$, μ, (X, d), (Y, ρ) is that given at the beginning. S will be a topological space.

Proposition 1.1. *Let Φ be a multifunction from S into Y. Then, the following assertions are equivalent*:

(*a*) *The multifunction Φ is lower semicontinuous.*

(*b*) *For every $y \in Y$, the real function $\rho(y, \Phi(\cdot))$ is upper semicontinuous.*

(*c*) *The set $\{y \in Y$: the real function $\rho(y, \Phi(\cdot))$ is upper semicontinuous$\}$ is dense in Y.*

Proof. For the implication $(a) \Rightarrow (b)$, we refer, for instance, to Theorem 1.1 of [10]. The implication $(b) \Rightarrow (c)$ is, of course, trivial. So, let (c) hold. Denote by D the set defined in (c). Let $t_0 \in S$ and let Ω be any open set in Y such that $\Phi(t_0) \cap \Omega \neq \emptyset$. Choose $y_0 \in \Phi(t_0) \cap \Omega$ and fix $\varepsilon > 0$ such that $B_\rho(y_0, \varepsilon) \subseteq \Omega$. Since D is dense in Y, there is $z_0 \in B_\rho(y_0, \varepsilon/2) \cap D$. Observe that $\rho(z_0, \Phi(t_0)) \leq \rho(z_0, y_0) < \varepsilon/2$. Since the real function $\rho(z_0, \Phi(\cdot))$ is upper semicontinuous, there is a neighbourhood U of t_0 such that $\rho(z_0, \Phi(t)) < \varepsilon/2$ for all $t \in U$. Therefore, $\Phi(t) \cap B_\rho(z_0, \varepsilon/2) \neq \emptyset$, and so $\emptyset \neq \Phi(t) \cap \cap B_\rho(y_0, \varepsilon) \subseteq \Phi(t) \cap \Omega$ for all $t \in U$. This proves (a). ■

Proposition 1.2. *Let Φ be a multifunction from S into Y such that the set $\{y \in Y$: the real function $\rho(y, \Phi(\cdot))$ is lower semicontinuous$\}$ is dense in Y. Then, the set $\{(t, y) \in S \times Y: y \in \overline{\Phi(t)}\}$ is closed in $S \times Y$.*

Proof. Put $E = \{y \in Y$: the real function $\rho(y, \Phi(\cdot))$ is lower semicontinuous$\}$. We shall prove that the set $\{(t, y) \in S \times Y: y \notin \overline{\Phi(t)}\}$ is open in $S \times Y$. Thus, fix $(t_0, y_0) \in S \times Y$ such that $y_0 \notin \overline{\Phi(t_0)}$. Hence, $\rho(y_0, \Phi(t_0)) > 0$. Fix $\varepsilon \in]0, \rho(y_0, \Phi(t_0))[$. Since E is dense in Y, there is $z_0 \in B_\rho(y_0, \varepsilon/4) \cap E$. Of course, we have $\rho(y_0, \Phi(t_0)) - - \varepsilon/4 \leqslant \rho(z_0, \Phi(t_0))$. Since the real function $\rho(z_0, \Phi(\cdot))$ is lower semicontinuous, there exists a neighbourhood U of t_0 such that $\rho(y_0, \Phi(t_0)) - \varepsilon/2 < \rho(z_0, \Phi(t))$ for all $t \in U$. Therefore, if $(t, y) \in U \times B_\rho(y_0, \varepsilon/4)$, we have $\rho(y, \Phi(t)) \geqslant \rho((z_0, \Phi(t)) - \rho(y, z_0) > > \rho(y_0, \Phi(t_0)) - \varepsilon > 0$. This completes the proof. ■

Proposition 1.3. *Let f be a real function defined on $T \times X$. Assume that*:

(*a*) *for every $t \in T$, the function $f(t, \cdot)$ is continuous;*

(*b*) *the set $\{x \in X$: the function $f(\cdot, x)$ is $\mathcal{F}$-measurable$\}$ is dense in X.*

Then, for every $\varepsilon > 0$ *there exists a compact set* $K_\varepsilon \subseteq T$, *with* $\mu(T \setminus K_\varepsilon) < \varepsilon$, *such that the function* $f|_{K_\varepsilon \times X}$ *is continuous.*

PROOF. First, observe that, since X is separable, by (b), there is a countable dense subset G of X such that, for every $x \in G$, the function $f(\cdot, x)$ is $\mathcal{F}$-measurable. Consider now the multifunction Ψ, from T into $X \times \mathbb{R}$, defined by putting, for every $t \in T$,

$$\Psi(t) = \{(x, r) \in X \times \mathbb{R} : f(t, x) \geq r\} .$$

We claim that Ψ is $\mathcal{F}$-measurable. To this end, let Ω be any open subset of X and $]\alpha, \beta[$ any open real interval. We have:

$$(1) \qquad \Psi^{-}(\Omega \times]\alpha, \beta[) = \{t \in T; \exists (x, r) \in \Omega \times]\alpha, \beta[: f(t, x) \geq r\} = \bigcup_{x \in \Omega} \{t \in T; f(t, x) > \alpha\} .$$

On the other hand, if $t_0 \in T$ is such that, for some $x_0 \in \Omega$, one has $f(t_0, x_0) > \alpha$, then, by (a), there is a neighbourhood U of x_0 such that $f(t_0, x) > \alpha$ for all $x \in U$. Since $\overline{G} = X$, there is $x^* \in \Omega \cap U \cap G$. So, in particular, $f(t_0, x^*) > \alpha$. In other words, we have:

$$(2) \qquad \bigcup_{x \in \Omega} \{t \in T : f(t, x) > \alpha\} = \bigcup_{x \in \Omega \cap G} \{t \in T : f(t, x) > \alpha\} .$$

Hence, by (1) and (2), we have $\Psi^{-}(\Omega \times]\alpha, \beta[) = \bigcup_{x \in \Omega \cap G} \{t \in T : f(t, x) > \alpha\}$. From this, by the properties of G, it follows that $\Psi^{-}(\Omega \times]\alpha, \beta[) \in \mathcal{F}$. Now, our claim is an immediate consequence of the fact that any open subset of $X \times \mathbb{R}$ is the union of a countable family of sets of the type $\Omega \times]\alpha, \beta[$, with Ω and $]\alpha, \beta[$, as above. By (a) again, $\Psi(t)$ is closed for all $t \in T$. Fix $\varepsilon > 0$. Then, by applying to the multifunction Ψ Lemma 2.2 of [1] (which holds also under our present assumptions on $\mathcal{F}$ and μ), we get a compact set $K'_\varepsilon \subseteq T$, with $\mu(T \setminus K'_\varepsilon) < \varepsilon/2$, such that the set $\{(t, x, r) \in K'_\varepsilon \times X \times \mathbb{R} : f(t, x) \geq r\}$ is closed in $K'_\varepsilon \times X \times \mathbb{R}$. So, in particular, for every $r \in \mathbb{R}$, the set $\{(t, x) \in K'_\varepsilon \times X : f(t, x) \geq r\}$ is closed in $K'_\varepsilon \times X$. That is, the function $f|_{K'_\varepsilon \times X}$ is upper semicontinuous. On the other hand, also the function $-f$ satisfies (a) and (b). Therefore, by what seen above, there is a compact set $K''_\varepsilon \subseteq T$, with $\mu(T \setminus K''_\varepsilon) < \varepsilon/2$, such that the function $-f|_{K''_\varepsilon \times X}$ is upper semicontinuous. Then, if we put $K_\varepsilon = K'_\varepsilon \cap K''_\varepsilon$, we have that $\mu(T \setminus K_\varepsilon) < \varepsilon$ and that the function $f|_{K_\varepsilon \times X}$ is continuous. ■

PROPOSITION 1.4. *Let f be a real function defined on $T \times X$. Assume that:*

(a) *for every $t \in T$, the function $f(t, \cdot)$ is upper semicontinuous;*

(b) *for every $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{F}$, with $\mu(T \setminus A_\varepsilon) < \varepsilon$, such that, for each $x \in X$, the function $f(\cdot, x)|_{A_\varepsilon}$ is lower semicontinuous.*

Then, for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq T$, with $\mu(T \setminus K_\varepsilon) < \varepsilon$, such that the function $f|_{K_\varepsilon \times X}$ is upper semicontinuous.

PROOF. The proof is similar to that of Proposition 1.3, and so, keeping the same notations of this latter, we give only a sketch. Fix $\varepsilon > 0$ and choose $A_\varepsilon \in \mathcal{F}$, with $\mu(T \setminus A_\varepsilon) < \varepsilon/2$, as in (b). Since $\Psi^{-}(\Omega \times]\alpha, \beta[) \cap A_\varepsilon = \bigcup_{x \in \Omega} \{t \in A_\varepsilon : f(t, x) > \alpha\}$, we infer easily that the multifunction $\Psi|_{A_\varepsilon}$ is lower semicontinuous, and so, a fortiori, it is $\mathcal{F}_{A_\varepsilon}$-measurable, where $\mathcal{F}_{A_\varepsilon} = \{\Gamma \in \mathcal{F} : \Gamma \subseteq A_\varepsilon\}$. Moreover, (a) ensures that $\Psi(t)$ is closed for all $t \in T$. By applying Lemma 2.2 of [1] to the multifunction $\Psi|_{A_\varepsilon}$, we get a compact set $K_\varepsilon \subseteq A_\varepsilon$, with $\mu(A_\varepsilon \setminus K_\varepsilon) < \varepsilon/2$, such that the set $\{(t,x,r) \in K_\varepsilon \times X \times \mathbb{R}: f(t,x) \geq r\}$ is closed in $K_\varepsilon \times X \times \mathbb{R}$. Hence, $\mu(T \setminus K_\varepsilon) < \varepsilon$ and $f|_{K_\varepsilon \times X}$ is upper semicontinuous. ■

2. Proofs of theorems 1 and 2

Let us begin with the proof of Theorem 1. Fix $\varepsilon > 0$. Since X is separable, by (*b*), there exists a countable dense subset C of X such that, for every $x \in C$, the multifunction $F(\cdot, x)$ is $\mathcal{F}$-measurable and the set $F(T, x)$ is separable. Then, by (*a*) and Proposition 2.2 of [9], the set $F(T \times X)$ turns out be separable. Thus, let $\{y_n\}$ be a dense sequence in $F(T \times X)$. For every $n \in \mathbb{N}$, $(t,x) \in T \times X$, put $f_n(t,x) = \rho(y_n, F(t,x))$. Fix $n \in \mathbb{N}$. From (*a*), taking into account Theorem 1.2 of [10] and Proposition 1.1, it follows that, for every $t \in T$, the function $f_n(t, \cdot)$ is continuous. Moreover, by (v) at p. 59 of [3], for every $x \in C$, the function $f_n(\cdot, x)$ is $\mathcal{F}$-measurable. Then, thanks to Proposition 1.3, there exists a compact set $K_{\varepsilon,n} \subseteq T$, with $\mu(T \setminus K_{\varepsilon,n}) < \varepsilon/2^n$, such that the function $f_n|_{K_{\varepsilon,n} \times X}$ is continuous. Now, put $K_\varepsilon = \bigcap_{n \in \mathbb{N}} K_{\varepsilon,n}$. So, $\mu(T \setminus K_\varepsilon) < \varepsilon$ and $f_n|_{K_\varepsilon \times X}$ is continuous for all $n \in \mathbb{N}$. Then, from Propositon 1.1 it follows that the multifunction $F|_{K_\varepsilon \times X}$ is lower semicontinuous, while Proposition 1.2 ensures that the set $\{(t,x,y) \in K_\varepsilon \times X \times Y: y \in \overline{F(t,x)}\}$ is closed in $K_\varepsilon \times X \times Y$. Thus, Theorem 1 is proved.

Let us prove now Theorem 2. From (*b*) it follows, in particular, that there exists a sequence $\{A_k\}$ in $\mathcal{F}$, with $\mu\left(T \setminus \bigcup_{k \in \mathbb{N}} A_k\right) = 0$, such that, for each $k \in \mathbb{N}$ and $x \in X$, the set $F(A_k, x)$ is separable. By (*a*) and Proposition 2.2 of [9], the set $F(A_k \times X)$ is separable. Hence, if we put $T^* = \bigcup_{k \in \mathbb{N}} A_k$, the set $F(T^* \times X)$ is separable. Let $\{z_n\}$ be a dense sequence in $F(T^* \times X)$. For every $n \in \mathbb{N}$, $(t,x) \in T^* \times X$, put $g_n(t,x) = \rho(z_n, F(t,x))$. Fix $n \in \mathbb{N}$. By (*a*) and Proposition 1.1, for every $t \in T^*$, the function $g_n(t, \cdot)$ is upper semicontinuous, while, by (*b*) and Theorem 1.2 of [10], for every $\varepsilon > 0$ we can find $A_\varepsilon \in \mathcal{F}$, with $A_\varepsilon \subseteq T^*$ and $\mu(T^* \setminus A_\varepsilon) < \varepsilon$, such that, for every $x \in X$, the function $g_n(\cdot, x)|_{A_\varepsilon}$ is lower semicontinuous. Then, thanks to Proposition 1.4, for every $\varepsilon > 0$ there exists a compact set $K_{\varepsilon,n} \subseteq T^*$, with $\mu(T^* \setminus K_{\varepsilon,n}) < \varepsilon/2^n$, such that the function $g_n|_{K_{\varepsilon,n} \times X}$ is upper semicontinuous. Put $K_\varepsilon = \bigcap_{k \in \mathbb{N}} K_{\varepsilon,n}$. So, $\mu(T \setminus K_\varepsilon) < \varepsilon$ and, for every $n \in \mathbb{N}$, the function $g_n|_{K_\varepsilon \times X}$ is upper semicontinuous. Hence, by Proposition 1.1, the multifunction $F|_{K_\varepsilon \times X}$ is lower semicontinuous. ■

3. Remarks and applications

First, we want to point out the following consequence of Theorem 1, whose simple proof is left to the reader.

Proposition 3.1. *Let μ be complete and Y be separable. Let F be a multifunction from $T \times X$ into Y such that, for every $t \in T$, the multifunction $F(t, \cdot)$ is continuous. Put $\Gamma = \{x \in X$: the multifunction $F(\cdot, x)$ is $\mathcal{F}$-measurable$\}$. If Γ is dense in X, then $\Gamma = X$.*

Theorem 1 is no longer true, even when Y is separable and, for every $x \in X$, the multifunction $F(\cdot, x)$ is $\mathcal{F}$-measurable, if one assumes only that, for every $t \in T$, the multifunction $F(t,\cdot)$ is lower semicontinuous. The reader can find a counterexample in this direction, for instance, at p. 546 of [1].

Consider now the following example.

Example 3.1. Let $T = X = Y = [0, 1]$ (equipped with the usual metric) and let $\mathcal{F}$ be the family of all Lebesgue measurable subsets of $[0, 1]$. Choose any not $\mathcal{F}$-measurable function $\varphi: [0, 1] \to [0, 1]$. For every $(t, x) \in [0, 1] \times [0, 1]$, put

$$F(t,x) = \begin{cases} [0,1] & \text{if } x > 0, \\ \{\varphi(t)\} & \text{if } x = 0. \end{cases}$$

Observe that, for every $t \in [0, 1]$, the multifunction $F(t, \cdot)$ is lower semicontinuous (but not continuous) and that, for every $x \in]0, 1]$, the multifunction $F(\cdot, x)$ is even continuous, being constant. However, since $F(\cdot, 0)$ is not $\mathcal{F}$-measurable, the conclusion of Theorem 2, in this case, does not hold.

Therefore, Example 3.1 shows that it is not possible to replace assumption (*b*) of Theorem 2 with the other «the set $\{x \in X$: the multifunction $F(\cdot, x)$ is upper semicontinuous$\}$ is dense in X», even if Y is separable.

Theorems 1 and 2 can be used in order to get Carathéodory's selections for a given multifunction. Here is a sample.

Theorem 3.1. *Let μ be complete. Let Y be a Banach space and let F be a multifunction from $T \times X$ into Y, with closed and convex values, satisfying conditions (a) and (b) of one of Theorems* 1 *and* 2. *Then, there exists a function f from $T \times X$ into Y, satisfying the following assertions:*

(i) $f(t, x) \in F(t, x)$ *for every* $(t, x) \in T \times X$;

(ii) *for every $t \in T$, the function $f(t, \cdot)$ is continuous;*

(iii) *for every $x \in X$, the function $f(\cdot, x)$ is $\mathcal{F}$-measurable.*

Proof. According to our assumptions, by one of Theorems 1 and 2, for every $n \in \mathbb{N}$ there exists a compact set $K_n \subseteq T$, with $\mu(T \setminus K_n) < 1/n$, such that the multifunction $F|_{K_n \times X}$ is lower semicontinuous. By a classical result (see, for instance, [2], p. 95, Proposition 17) the space $K_n \times X$ is paracompact. Hence, by the classical continuous selection theorem of Michael ([8], Theorem 3.2″) there is a continuous function $f_n: K_n \times X \to Y$ such that $f_n(t, x) \in F(t, x)$ for every $(t, x) \in K_n \times X$. Always by Michael's theorem, for each $t \in T \setminus \bigcup_{n \in \mathbb{N}} K_n$, we can choose a continuous selection ψ_t of the multifunction $F(t, \cdot)$. Now, for each $(t, x) \in T \times X$, put

$$f(t,x) = \begin{cases} f_1(t,x) & \text{if } t \in K_1 \\ f_n(t,x) & \text{if } t \in K_n \setminus \bigcup_{j=1}^{n-1} K_j, \quad n \geq 2 \\ \psi_t(x) & \text{if } t \in T \setminus \bigcup_{n \in \mathbb{N}} K_n. \end{cases}$$

Taking into account the completeness of μ, it is immediate to check that the function f satisfies (i), (ii) and (iii). ■

Observe that Theorem 3.1 improves Theorem 1 of [4], provided conditions (*a*) and (*b*) of Theorem 2 hold. The main improvement resides in the fact that we do not assume the compactness of the values of F.

For other papers on the Scorza Dragoni property for multifunctions, we refer to [1], [5], [6], [7], [12]. Observe, in particular, that in each of these papers the metric space X is assumed to be also complete: an assumption we do not need.

Observe, finally, that Theorem 1 extends to multifunctions Theorem 1 of [11].

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