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A uniqueness criterion for the solution of the stationary Navier-Stokes equations

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Analisi matematica. — *A uniqueness criterion for the solution of the stationary Navier-Stokes equations* (*). Nota (**) del Corrisp. GIOVANNI PROUSE.

ABSTRACT. — A uniqueness criterion is given for the weak solution of the Navier-Stokes equations in the stationary case. Precisely, it is proved that, for a generic known term, there exists one and only one solution such that the mechanical power of the corresponding flow is maximum and that this maximum is «stable» in an appropriate sense.

KEY WORDS: Fluid dynamics; Weak solution; Analytic function.

RIASSUNTO. — *Un criterio di unicità per la soluzione delle equazioni di Navier-Stokes nel caso stazionario.* Si dà un criterio di unicità per la soluzione debole delle equazioni di Navier-Stokes nel caso stazionario. Precisamente si dimostra che, per un termine noto generico, esiste una ed una sola soluzione tale che la potenza meccanica relativa al moto da essa individuato sia massima e che tale massimo sia «stabile» in senso opportuno.

1. It is well-known that no uniqueness theorem has yet been proved for the solutions of the stationary Navier-Stokes equations

$$\mu \Delta \boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla p - \boldsymbol{f} = 0, \qquad \nabla\cdot\boldsymbol{u} = 0 \tag{1.1}$$

in an open, bounded set Ω of R^m $(m = 2, 3)$, with the boundary condition

$$\boldsymbol{u}|_{\partial\Omega} = 0, \tag{1.2}$$

except in special cases in which the viscosity coefficient μ is «large» or the external force is «small».

It is equally well-known (see, for instance, [1]) that, if $\boldsymbol{f} \in H^{-1}$, there exists $\boldsymbol{u} \in N^1$ (¹), $p \in L^1$ such that (1.1), (1.2) are satisfied in the sense of distributions on Ω, or, equivalently, that

$$\mu(\boldsymbol{u}, \boldsymbol{\phi})_{H_0^1} + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{\phi})_{L^2} = \langle \boldsymbol{f}, \boldsymbol{\phi}\rangle \qquad \forall \boldsymbol{\phi} \in N^1. \tag{1.3}$$

The aim of the present paper is to show that, *for a generic external force* $\boldsymbol{f}$, *among all the solutions of the problem considered, there exists one, and only one,* $\tilde{\boldsymbol{u}}$, *such that the mechanical power* $(\boldsymbol{f}, \boldsymbol{u})_{L^2}$ *relative to the stationary flow described by* $\tilde{\boldsymbol{u}}$ *is maximum and that this maximum is «stable» in an appropriate sense.*

Let $\{g_j\}$, $\{\lambda_j\}$ be the eigenfunctions (normalized in L^2) and the eigenvalues of the operator $-\Delta$, from H_0^1 to H^{-1}; the set $\{g_j\}$ is then a basis in H_0^1 and in L^2 and

$$\left(\frac{g_j}{\sqrt{\lambda_j}}, \frac{g_k}{\sqrt{\lambda_k}}\right)_{H_0^1} = (g_j, g_k)_{L^2} = \delta_{jk}, \tag{1.4}$$

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots, \qquad \lim_{n\to\infty} \lambda_n = +\infty. \tag{1.5}$$

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(¹) N^1 is defined as the closure in H^1 of the space $N = \{v \in \mathcal{D}, \nabla\cdot v = 0\}$; the symbol $\langle\, ,\rangle$ will, in what follows, denote the duality between H^{-1} and H_0^1.

Setting now, for $u \in H_0^1$, $0 \leq \theta \leq 1/2$,

$$L_\theta(u) = \langle (-\Delta)^\theta f, u \rangle = (f, (-\Delta)^\theta u)_{L^2}, \tag{1.6}$$

we shall say that *the solution $\tilde{u}$ is maximal θ-stable if, denoting by u any other solution, there exists a neighbourhood $(0, \theta_u)$ of $\theta = 0$* (θ_u depending on u) *such that*

$$L_\theta(\tilde{u}) > L_\theta(u) \qquad \forall \theta \in (0, \theta_u). \tag{1.7}$$

Setting

$$\begin{cases} f = \sum_{j=1}^{\infty} \phi_j g_j, & \phi_j = (f, g_j)_{L^2}, \quad \sum_{j=1}^{\infty} \phi_j^2 < +\infty, \\ u = \sum_{j=1}^{\infty} \alpha_j g_j, & \alpha_j = (u, g_j)_{L^2}, \quad \sum_{j=1}^{\infty} \lambda_j \alpha_j^2 < +\infty, \end{cases} \tag{18}$$

we also have, by (1.4) (see, for instance, [2])

$$L_\theta(u) = (f, (-\Delta)^\theta u)_{L^2} = \left(\sum_{j=1}^{\infty} \phi_j g_j, \sum_{k=1}^{\infty} \lambda_k^\theta \alpha_k g_k \right)_{L^2} = \sum_{j=1}^{\infty} \lambda_j^\theta \alpha_j \phi_j. \tag{1.9}$$

The theorem we shall prove is the following.

Assume that $f \in L^2$ and does not belong to any subspace spanned by a subsequence of the $\{g_j\}$ (i.e. $(f, g_j)_{L^2} \neq 0 \ \forall j$); *there exists then one, and only one, solution $\tilde{u}$ which is maximal θ-stable.*

The necessity of the condition set on f is obvious; if, in fact, it were $\phi_p = (f, g_p)_{L^2} = 0$, the p-th term in the expansion (1.9) would be missing and the coefficient α_p would consequently remain undetermined.

The proof of the theorem stated above will be given in section 3, while some auxiliary lemmas will be proved in the next section.

2. We now prove some auxiliary lemmas, always assuming that $f \in L^2$, $u \in H_0^1$.

LEMMA 1. *Let U denote the set of solutions, i.e. of functions $u \in N^1$ satisfying* (1.3). *U is compact in H_0^s, $\forall s < 1$.*

Setting, in fact, in (1.3), $\phi = u$, we obtain directly, since $((u \cdot \nabla) u, u)_{L^2} = 0$,

$$\|u\|_{H_0^1} \leq M. \tag{2.1}$$

Let $\{u_n\}$ be a sequence $\subset U$; by (2.1) and a well-known compactness theorem, it is possible to select from $\{u_n\}$ a subsequence $\{u_{n'}\}$ such that

$$\lim_{n' + \infty} u_{n'} = u \tag{2.2}$$

in the weak topology of H_0^1 and in the strong topology of $H_0^s (s < 1)$. Bearing in mind that $((u \cdot \nabla) u, \phi)_{L^2} = -((u \cdot \nabla) \phi, u)_{L^2}$, by (1.3), (2.2) it follows that u is also a solution.

LEMMA 2. *The function $\theta \rightarrow L_0(u)$ can be expanded in the power series*

$$L_\theta(u) = \sum_{j=1}^{\infty} \lambda_j^\theta \alpha_j \phi_j = \sum_{k=0}^{\infty} \frac{\partial^k L_\theta(u)}{\partial \theta^k} \bigg|_{\theta = 0} \frac{\theta^k}{k!} = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \sum_{j=1}^{\infty} (\log \lambda_j)^k \alpha_j \phi_j \tag{2.3}$$

in the neighbourhood $|\theta| < 1/2$ of $\theta = 0$

Consider, in fact, the power series in the complex variable z

$$\sum_{j=1}^{\infty} \lambda_j^z \alpha_j \phi_j ; \tag{2.4}$$

we have, $\forall z$ with $|z| < 1/2$ and $\bar{j}$ sufficiently large, bearing in mind that $\lim\limits_{j\to\infty} \lambda_j = +\infty$,

$$\Big|\sum_{j=\bar{j}}^{\infty} \lambda_j^z \alpha_j \phi_j\Big| \le \sum_{j=\bar{j}}^{\infty} \lambda_j^{|z|} |\alpha_j|\, |\phi_j| \le \sum_{j=\bar{j}}^{\infty} \lambda_j^{1/2} |\alpha_j|\, |\phi_j| \le \|f\|_{L^2} \|u\|_{H_0^1} . \tag{2.5}$$

Hence, by the Weierstrass criterion, the series (2.4) converges uniformly in every closed domain with $|z| < 1/2$ and represents therefore an analytic function, holomorphic for $|z| < 1/2$. It follows directly that expansion (2.3) holds.

Lemma 3. *Consider, for $k = 1, 2, \ldots$, the series*

$$\sum_{j=1}^{\infty} (\log \lambda_j)^{k-1} \phi_j g_j . \tag{2.6}$$

These series converge to elements $y_k \in H^{-\varepsilon}$ $\forall \varepsilon > 0$:

$$y_k = \sum_{j=1}^{\infty} (\log \lambda_j)^{k-1} \phi_j g_j \in H^{-\varepsilon} . \tag{2.7}$$

We recall that the series $\sum\limits_{j=1}^{\infty} \gamma_j g_j$ converges an element of H^s if $\sum\limits_{j=1}^{\infty} \lambda_j^s \gamma_j^2 < +\infty$.

Observe, moreover, that, since $\lambda_j \to \infty$ when $j \to \infty$, there exists, $\forall$ fixed $\varepsilon = 0$, a constant M_ε such that

$$|\log \lambda_j| \le M_\varepsilon + \lambda_j^\varepsilon . \tag{2.8}$$

We have then, $\forall$ fixed k and $\forall \varepsilon > 0$,

$$\sum_{j=1}^{\infty} \lambda_j^{-\varepsilon} ((\log \lambda_j)^{k-1} \phi_j)^2 \le \sum_{j=1}^{\infty} \lambda_j^{-\varepsilon} (M_{k,\varepsilon} + \lambda_j^{\varepsilon/2})^2 \phi_j^2 \le 2 M_{k,\varepsilon} \|f\|^2_{H^{-\varepsilon}} + 2 \|f\|^2_{L^2} \le M'_{k,\varepsilon} . \tag{2.9}$$

This proves our lemma.

Lemma 4. *Let $\{g_j^*\}$ be a subsequence of $\{g_j\}$ such that the corresponding eigenvalues $\{\lambda_j^*\}$ are all different and denote by $H_*^{-\varepsilon}$ the subspace of $H^{-\varepsilon}$ spanned by $\{g_j^*\}$, i.e.*

$$z \in H_*^{-\varepsilon} \Leftrightarrow z = \sum_{j=1}^{\infty} \zeta_j g_j^* , \ \zeta_j = \langle z, g_j^* \rangle , \ \sum_{j=1}^{\infty} \lambda_j^{*-\varepsilon} \zeta_j^2 < +\infty .$$

Setting

$$y_k^* = \sum_{j=1}^{\infty} (\log \lambda_j^*)^{k-1} \phi_j^* g_j^* \qquad (\phi_j^* = (f, g_j^*)_{L^2},\ k = 1, 2, \ldots) , \tag{2.10}$$

the sequence $\{y_k^*\}$ is a basis in $H_*^{-\varepsilon}$, $\forall \varepsilon > 0$.

Let us show, to begin with, that, $\forall$ fixed n, it is possible to express $g_1^*, \ldots, g_n^*$ as linear combinations of $y_1^*, \ldots, y_n^*$:

$$g_j^* = \sum_{k=1}^{\infty} \eta_{jk}^{(n)} y_k^* \qquad (j = 1, \ldots, n) . \tag{2.11}$$

Multiplying (2.11) by g_p^* we obtain

$$\delta_{jp} = (g_j^*, g_p^*)_{L^2} = \sum_{k=1}^{n} \eta_{jk}^{(n)} (y_k^*, g_p^*)_{L^2} = \sum_{k=1}^{n} \eta_{jk}^{(n)} \left(\sum_{i=1}^{\infty} (\log \lambda_i^*)^{k-1} \phi_i^* g_i^*, g_p^* \right)_{L^2} = \tag{2.12}$$

$$= \sum_{k=1}^{n} \eta_{jk}^{(n)} (\log \lambda_p^*)^{k-1} \phi_p^* = \sum_{k=1}^{n} \alpha_{pk} \eta_{jk}^{(n)} \qquad (j, p = 1, \dots, n).$$

For fixed j, (2.12) represent a linear system of equations in the unknowns $\eta_{j1}^{(n)}, \dots, \eta_{jn}^{(n)}$; this system can be uniquely solved, since its determinant

$$\det[\alpha_{pk}] = \det[\phi_p^* (\log \lambda_p^*)^{k-1}] \tag{2.13}$$

does not vanish, by the assumptions that $\phi_p^* \neq 0$ $\forall p$ and that all the λ_p^*'s are different. Relations (2.11) are therefore proved.

Let now $z = \sum_{j=1}^{\infty} \zeta_j g_j^*$ be an arbitrary element of $H_*^{-\varepsilon}$; we have obviously,

$$\lim_{n+\infty} \| z - \sum_{j=1}^{n} \zeta_j g_j^* \|_{H_*^{-\varepsilon}} = 0. \tag{2.14}$$

On the other hand, by (2.11),

$$\sum_{j=1}^{n} \zeta_j g_j^* = \sum_{j=1}^{n} \zeta_j \sum_{k=1}^{n} \eta_{jk}^{(n)} y_k^* = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \zeta_j \eta_{jk}^{(n)} \right) y_k^* = \sum_{k=1}^{n} \sigma_k^{(n)} y_k^*. \tag{2.15}$$

Hence, by (2.14), (2.15), the coefficients $\sigma_k^{(n)} = \sum_{j=1}^{n} \zeta_j \eta_{jk}^{(n)}$, with $\eta_{jk}^{(n)}$ solutions of (2.12), are such that

$$\lim_{n \to \infty} \| z - \sum_{k=1}^{n} \sigma_k^{(n)} y_k^* \|_{H_*^{-\varepsilon}} = 0. \tag{2.16}$$

This proves that $\{y_k^*\}$ is basis in $H_*^{-\varepsilon}$.

LEMMA 5. *The set* (defined in lemma 3)

$$y_k = \sum_{j=1}^{\infty} (\log \lambda_j)^{k-1} \phi_j g_j \in H^{-\varepsilon} \qquad (k = 1, 2, \dots) \tag{2.17}$$

is a basis in $H^{-\varepsilon}$, $\forall \varepsilon > 0$.

Since all the eigenvalues λ_j have finite multiplicity, it is obviously possible to select from the sequence of eigenfunctions $\{g_j\}$ a set of sequences $\{g_j^{(\alpha)}\}$ $(\alpha = 1, 2, \dots)$ such that in each sequence $\{g_j^{(\alpha)}\}$ the corresponding eigenvalues $\{\lambda_j^{(\alpha)}\}$ are all simple. Denoting by $H_\alpha^{-\varepsilon}$ the subspace of $H^{-\varepsilon}$ spanned by $\{g_j^{(\alpha)}\}$, we have moreover,

$$H^{-\varepsilon} = H_1^{-\varepsilon} \oplus H_2^{-\varepsilon} \oplus \dots. \tag{2.18}$$

Since, by lemma 4,

$$y_k^{(\alpha)} = \sum_{j=1}^{\infty} (\log \lambda_j^{(\alpha)})^{k-1} \phi_j^{(\alpha)} g_j^{(\alpha)} \qquad (\phi_j^{(\alpha)} = (f, g_j^{(\alpha)})_{L^2}, k = 1, 2 \dots) \tag{2.19}$$

is a basis in $H_\alpha^{-\varepsilon}$, it follows directly from (2.19) that $\{y_k\}$ is a basis in $H^{-\varepsilon}$.

3. Let us now prove the theorem stated in section 1. Consider, at first, the functional $u = L_0(u) = (f, u)_{L^2}$, $f \in L^2$, $u \in U$; this functional is obviously H_0^s-continuous, $\forall s \leq 1$. Since, by lemma 1, the set U of solutions is H_0^s-compact $(s < 1)$, there

exists $u_1 \in U$ such that

$$L_0(u_1) \geq L_0(u) \qquad \forall u \in U. \tag{3.1}$$

Setting

$$\sigma_1 = L_0(u_1), \tag{3.2}$$

we denote by U_1 the subset of U constituted by the solutions u such that

$$L_0(u) = \sigma_1. \tag{3.3}$$

The set U_1 is obviously $\neq \emptyset$ and is H_0^s-compact, $\forall s < 1$. In fact, by lemma 1, if $\{u_n\}$ is a sequence $\in U_1$ with $u_n \to u$ in H_0^1, then $u \in U$; it is obvious, moreover, that $L_\theta(u_n) = (f, u_n)_{L^2} \to (f, u)_{L^2} = L_0(u) = \sigma_1$.

Consider now the functional

$$u \to \left.\frac{\partial L_0(u)}{\partial \theta}\right|_{\theta=0} = \sum_{j=1}^{\infty} (\log \lambda_j)\, \alpha_j \phi_j. \tag{3.4}$$

We have, bearing in mind (2.8),

$$\begin{aligned} \left|\sum_{j=1}^{\infty} (\log \lambda_j)\, \phi_j \alpha_j\right| \leq \sum_{j=1}^{\infty} (M_{1/2} + \lambda_j^{1/2}) |\phi_j|\, |\alpha_j| \leq M_{1/2} \|f\|_{L^2} \|u\|_{L^2} + \\ + \|f\|_{L^2} \|u\|_{H_0^1} \leq C_1 \|f\|_{L^2} \|u\|_{H_0^1}, \end{aligned} \tag{3.5}$$

and, consequently, $(\partial L_0(u)/\partial\theta)|_{\theta=0}$ is H_0^s-continuous $\forall s \leq 1$ and $\forall u \in U$.

Since U_1 is H_0^s-compact $\forall s < 1$, there exists $u_2 \in U_1$ such that

$$\sigma_2 = \left.\frac{\partial L_\theta(u_2)}{\partial \theta}\right|_{\theta=0} \geq \left.\frac{\partial L_\theta(u)}{\partial \theta}\right|_{\theta=0} \qquad \forall u \in U_1. \tag{3.6}$$

We then denote by U_2 the H_0^s-compact set $\subset U_1$ constituted by the solutions u such that

$$\left.\frac{\partial L_\theta(u)}{\partial \theta}\right|_{\theta=0} = \sigma_2 \tag{3.7}$$

and repeat this procedure for the functionals

$$u \to \left.\frac{\partial^k L_\theta(u)}{\partial \theta^k}\right|_{\theta=0} = \sum_{j=1}^{\infty} (\log \lambda_j)^k \phi_j \alpha_j \qquad (k = 2, 3, \ldots). \tag{3.8}$$

We obtain in this way a monotonic decreasing sequence of H_0^s-compact sets $\neq \emptyset$, defined by

$$U_{k+1} = \left\{ u \in U_k;\ \left.\frac{\partial^k L_\theta(u)}{\partial \theta^k}\right|_{\theta=0} = \sigma_{k+1} \right\} \subset U_k. \tag{3.9}$$

with

$$\sigma_{k+1} = \max_{u \in U_k} \left.\frac{\partial^k L_\theta(u)}{\partial \theta^k}\right|_{\theta=0}, \qquad U_0 = U. \tag{3.10}$$

Let

$$\widetilde{U} = \lim_{k \to \infty} U_k; \tag{3.11}$$

it is evident, by definition, that $\widetilde{U} \neq \emptyset$; moreover, if $\tilde{\boldsymbol{u}} \in \widetilde{U}$, $\tilde{\boldsymbol{u}}$ is a solution such that

$$\left.\frac{\partial^k L_\theta(\tilde{\boldsymbol{u}})}{\partial \theta^k}\right|_{\theta=0} = \sigma_{k+1} \qquad \forall k = 0, 1, \dots . \tag{3.12}$$

Let us show that $\widetilde{U}$ is constituted by a single element, $\tilde{\boldsymbol{u}}$, i.e. if $\boldsymbol{u}, \boldsymbol{v} \in \widetilde{U}$, then $\boldsymbol{u} = \boldsymbol{v}$. We have, in fact, in this case

$$\left.\frac{\partial^k L_\theta(\boldsymbol{u})}{\partial \theta^k}\right|_{\theta=0} = \left.\frac{\partial^k L_\theta(V)}{\partial \theta^k}\right|_{\theta=0} = \sigma_{k+1} \qquad (k = 0, 1, \dots) \tag{3.13}$$

and, by (3.8), setting $\alpha_j = (\boldsymbol{u}, \boldsymbol{g}_j)_{L^2}$, $\beta_j = (\boldsymbol{v}, \boldsymbol{g}_j)_{L^2}$, $\gamma_j = \alpha_j - \beta_j$,

$$\sum_{j=1}^{\infty} (\log \lambda_j)^k \phi_j \gamma_j = 0 \qquad (k = 0, 1, \dots) . \tag{3.14}$$

Equations (3.14) can be written in the form

$$\langle \boldsymbol{y}_k, \boldsymbol{w} \rangle = 0 \qquad (k = 1, 2, \dots) \tag{3.15}$$

where $\boldsymbol{y}_k$ is given by (2.17) and $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v} = \sum_{j=1}^{\infty} \gamma_j \boldsymbol{g}_j$.

Since, by lemma 5, $\{\boldsymbol{y}_k\}$ is a basis in H^{-1} and $\boldsymbol{w} \in H_0^1$, it follows that $\boldsymbol{w} = 0$.

Finally, we prove that $\tilde{\boldsymbol{u}}$ is the only maximal θ-stable solution. Denoting, in fact, by $\boldsymbol{u}$ any other solution (necessarily $\notin \widetilde{U}$), there exists $p \geq 0$ such that

$$\left.\frac{\partial^p L_\theta(\boldsymbol{u})}{\partial \theta^p}\right|_{\theta=0} > \left.\frac{\partial^p L_\theta(\boldsymbol{u})}{\partial \theta^p}\right|_{\theta=0} , \tag{3.16}$$

while, if $p \geq 1$,

$$\left.\frac{\partial^k L_\theta(\tilde{\boldsymbol{u}})}{\partial \theta^k}\right|_{\theta=0} = \left.\frac{\partial^k L_\theta(\boldsymbol{u})}{\partial \theta^k}\right|_{\theta=0} \qquad \text{for } k = 0, \dots, p-1 . \tag{3.17}$$

Bearing in mind the expansion (2.3), it follows then that

$$L_\theta(\tilde{\boldsymbol{u}}) > L_\theta(\boldsymbol{u}) \tag{3.18}$$

in an appropriate neighbourhood $(0, \theta_u')$ of $\theta = 0$.

The theorem is thus proved.

Remark. The result obtained can be extended to more general abstract equations and to functionals which generalize (1.6). Let, in fact, V and H be two Hilbert spaces, with $V \subset H$, dense in H and with compact embedding in H; we shall identify H with its dual H' and denote by V' the dual of V. There exists then a sequence of elements $\{g_j\} \in V$ and a sequence of numbers $\{\lambda_j\}$ such that $\{g_j\}$ is basis in V and H and

$$(g_j, \phi)_V = \lambda_j (g_j, \phi)_H , \tag{3.19}$$

$$(g_j, g_k)_H = \delta_{jk} , \tag{3.20}$$

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots , \qquad \lim_{n \to \infty} \lambda_n = +\infty . \tag{3.21}$$

Consider the abstract equations

$$Au = f \tag{3.22}$$

where A is an operator from V to V' and f is a given element of a Hilbert space $W \subseteq V'$ and let T_θ $(0 \leq \theta \leq \bar{\theta})$ be a family of bounded operators from W to V'. We define, $\forall u \in V$, the functional

$$L_\theta(u) = \langle T_\theta f, u \rangle \tag{3.23}$$

and assume that it satisfies the following conditions

$$u \to L_\theta(u) \text{ is linear and } V\text{-continuous } \forall \theta \in [0, \bar{\theta}], \tag{3.24}$$

$$u \to L_\theta(u) \text{ is analytic, holomorphic for } |\theta| < \theta' \leq \bar{\theta}. \tag{3.25}$$

Since, by the assumption made, $\{g_j\}$ is a basis also in V', we can set

$$T_\theta f = \sum_{j=1}^{\infty} \psi_{f,\theta,j} g_j, \qquad \psi_{f,\theta,j} = \langle T_\theta f, g_j \rangle, \qquad \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \psi^2_{f,\theta,j} < +\infty \tag{3.26}$$

and, consequently, setting $\alpha_j = (u, g_j)_{L^2}$, by (3.25),

$$L_\theta(u) = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \left. \frac{\partial^k L_\theta(\boldsymbol{u})}{\partial \theta^k} \right|_{\theta=0} = \sum_{j=1}^{\infty} \psi_{f,\theta,j} \alpha_j, \tag{3.27}$$

where

$$\left. \frac{\partial^k L_\theta(\boldsymbol{u})}{\partial \theta^k} \right|_{\theta=0} = \sum_{j=1}^{\infty} \frac{\partial^k \psi_{f,0,j}}{\partial \theta^k} \alpha_j. \tag{3.28}$$

The following theorem then holds.

Assume that the set U of solutions of (3.22) *corresponding to f is compact and that the set* $\{h_k\}$ *defined by*

$$h_k = \sum_{j=1}^{\infty} \frac{\partial^k \psi_{f,0,j}}{\partial \theta^k} g_j. \qquad (k = 0, 1, \ldots) \tag{3.29}$$

is a basis in V'. *There exists then at most one solution of* (3.22) *which is maximal* θ*-stable.*

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