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Bounce trajectories in plane tubular domains

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Analisi matematica. — *Bounce trajectories in plane tubular domains.* Nota (*) di ROBERTO PEIRONE, presentata dal Socio. E. DE GIORGI.

ABSTRACT. — We state that in opportune tubular domains any two points are connected by a bounce trajectory and that there exist non-trivial periodic bounce trajectories.

KEY WORDS: Tubular domains; Bounce trajectories; Illumination of domains.

RIASSUNTO. — *Traiettorie di rimbalzo in domini tubolari piani.* Si stabilisce che in opportuni domini tubolari piani due punti qualunque sono congiungibili da una traiettoria di rimbalzo e inoltre esistono traiettorie periodiche di rimbalzo non banali.

In this note we study some problems concerning bounce trajectories, i.e. the motion of a point in a region $\overline{\Omega} \subseteq \boldsymbol{R}^n$, bouncing against $\partial\Omega$. We consider the following two problems:

1) Given two points in $\overline{\Omega}$, is there a bounce trajectory connecting them?

2) Are there periodic bounce trajectories of some particular kind?

With regard to problem 1) we remark that it has the following interpretation: Let us suppose that $\partial\Omega$ is a reflecting mirror. Does a light source in any point of $\overline{\Omega}$ illuminate all $\overline{\Omega}$?

We also remark that Problem 1) has a negative answer if Ω is (for example) the region between two circles C_1 and C_2 such that C_2 lies in the interior of C_1 and the centre A of C_1 belongs to C_2, for no bounce trajectory connects A to points lying in the open semiplane determined by the tangent to C_2 at A and containing C_1.

A more complicated example due to L. Penrose, R. Penrose (see [8], [6]), based on an ellipse, shows that there exists a region $\Omega \subseteq \boldsymbol{R}^2$ such that there exist two points in Ω which are not connected by any bounce trajectory. (In the preceding example one of the points was in $\partial\Omega$). In this note we consider non-convex regions $\Omega \subseteq \boldsymbol{R}^2$ in which Problem 1) has an affirmative answer.

We shall use the following notation:

d denotes the euclidean metric in $\boldsymbol{R}^2$ and $\| \ \|$ denotes the euclidean norm in $\boldsymbol{R}^2$.

If $A, B \in \boldsymbol{R}^2$ we denote by $[A, B]$ the set $\{tA + (1-t)B : 0 \le t \le 1\}$ and by $]A, B[$ the set $\{tA + (1-t)B : 0 < t < 1\}$.

If $A_1, \ldots, A_n$ are points in $\boldsymbol{R}^2$ we denote by $[A_1, \ldots, A_n]$ both the set $\bigcup_{1 \le i \le n-1} [A_i, A_{i+1}]$ and the function from $[0,1]$ into $\boldsymbol{R}^2$ which takes the value A_m at

$$\frac{\sum_{i=1}^{m-1} \|A_{i+1} - A_i\|}{\sum_{i=1}^{n-1} \|A_{i+1} - A_i\|}$$

(*) Pervenuta all'Accademia il 28 settembre 1987 for $m = 1, \dots, n$ and is linear in each interval

$$\left[\frac{\sum_{i=1}^{m-1}\|A_{i+1}-A_i\|}{\sum_{i=1}^{n-1}\|A_{i+1}-A_i\|}, \frac{\sum_{i=1}^{m}\|A_{i+1}-A_i\|}{\sum_{i=1}^{n-1}\|A_{i+1}-A_i\|}\right].$$

Given a region Ω in $\boldsymbol{R}^2$ with boundary $\partial\Omega$ of class C^1, we say that a continuous curve $\alpha: [0,1] \to \overline{\Omega}$ is a (classical) bounce trajectory (in Ω) if it has the form $\alpha = [A_1, A_2, \dots, A_n]$ and

i) $A_i \in \partial\Omega$ if $1 < i < n$.

ii) $]A_i, A_{i+1}[\subseteq \Omega$ if $1 < i < n$.

iii) The normal to $\partial\Omega$ at A_i bisects the angle $A_{i-1}\hat{A}_i A_{i+1}$ if $1 < i < n$.

In such a case we say that α connects A_1 to A_n and that $A_2, \dots, A_{n-1}$ are the bounce points of α. We say that α is a (classical) periodic bounce trajectory (in Ω) if $A_1 = A_n$ and $[A_1, A_2, \dots, A_n, A_2]$ is a bounce trajectory (non-classical bounce trajectories are continuous curve from $[0,1]$ into $\overline{\Omega}$ that in some cases can «run along $\partial\Omega$»; for a precise definition see [5]). The solutions we give to Problems 1) and 2) are classical bounce trajectories.

Let $\gamma_0: \boldsymbol{R} \to \boldsymbol{R}^2$ be a Jordan curve of class C^4, in the sense that γ_0 is a function of class C^4, $\gamma_0'(t) \neq (0,0)$ for every $t \in \boldsymbol{R}$ and $\gamma_0(a) = \gamma_0(b)$ if and only if $a - b \in \boldsymbol{Z}$; we put $D_0 = \{\gamma_0(t): t \in \boldsymbol{R}\}$. Let $\bar{\varepsilon}$ be a positive real number such that for any $z \in \boldsymbol{R}^2$ for which $d(z, D_0) < \bar{\varepsilon}$, there exists a unique $p_0(z) \in D_0$ such that $d(z, p_0(z)) < \bar{\varepsilon}$ and $[z, p_0(z)]$ is orthogonal to γ_0 at $p_0(z)$. Let us fix $\varepsilon < \bar{\varepsilon}$ and define

$$\Omega_\varepsilon = \{z \in \boldsymbol{R}^2 \mid d(z, D_0) < \varepsilon\}.$$

We remark that, denoting by E_1 and E_2 the connected components of $\boldsymbol{R}^2 \setminus D_0$, and putting $D_{(-1)^i\varepsilon} = \{z \in E_i \mid d(z, D_0) = \varepsilon\}$, we have $\partial\Omega_\varepsilon = D_\varepsilon \cup D_{-\varepsilon}$ and $D_{(-1)^i\varepsilon}$ has a representation as a Jordan curve of class C^3

$$\gamma_{(-1)^i\varepsilon} = p_0^{-1} \circ \gamma_0 . \qquad (i = 1, 2).$$

We say that Ω_ε is a tubular domain and γ_0 is its central curve.

In Ω_ε Problem 1) has an affirmative answer. Moreover, given two points in $\overline{\Omega}_\varepsilon$ and $N \in \boldsymbol{Z}$, there exists a bounce trajectory connecting them, with direction arbitrarily close to the interior normal in any its bounce point and winding N times. We can state the preceding concepts precisely in the following way.

If $a \in \{-\varepsilon, \varepsilon\}$, $A \in D_a$, $B \in D_{-a}$, $]A, B[\subseteq \Omega_\varepsilon$, then we denote by $\vartheta(A, B)$ the angle between $[A, B]$ and the interior normal to γ_a at A. We determine the sign of $\vartheta(A, B)$ according to the following convention: Let τ be a continuous function from $[0,1]$ into $\boldsymbol{R}$ such that $p_0(tA + (1-t)B) = \gamma_0(\tau(t))$. Then $\vartheta(A, B)$ is considered positive if τ is decreasing in $[0,1]$ and negative if τ is increasing in $[0,1]$.

If α is a continuous function from $[0,1]$ into $\overline{\Omega}_\varepsilon$, then we define $l_0(\alpha)$ (the lenght of $p_0 \circ \alpha$), to be

$$\int_{\tau(0)}^{\tau(1)} \|\gamma_0'(t)\| \, dt$$

provided that τ is a continuous function from $[0,1]$ into $\boldsymbol{R}$ such that $p_0 \circ \alpha = \gamma_0 \circ \tau$. Then, if we define $l_0(A,B)$ to be

$$\min\left\{ \int_u^v \|\gamma_0'(t)\|\,dt\colon\ \gamma_0(u) = p_0(A), \gamma_0(v) = p_0(B), \int_u^v \|\gamma_0'(t)\|\,dt \geq 0 \right\},$$

it is easily seen that, if $p_0(\alpha(0)) = p_0(A)$, then $p_0(\alpha(1)) = p_0(B)$ if and only if $(l_0(\alpha) - l_0(A,B))/L_0 \in \boldsymbol{Z}$ where L_0 denotes the lenght of γ_0.

We now may state precisely our result:

1. THEOREM. *For any two points A, B in $\overline{\Omega}_\varepsilon$, for any $\eta > 0$ and for any $N \in \boldsymbol{Z}$ there exists a bounce trajectory $\alpha = [z_1, z_2, \ldots, z_{n-1}, z_n]$ in Ω_ε connecting A to B such that*

$$\max\left(|\vartheta(z_i, z_{i+1})|, |\vartheta(z_i, z_{i-1})|\right) < \eta \qquad \textit{for } i = 2, \ldots, n-1$$

$$l_0(\alpha) = NL_0 + l_0(A,B) \qquad \textit{and}$$

$$\textit{if } z_i \in D_a \textit{ then } z_{i+1} \in D_{-a} \qquad \textit{for } i = 2, \ldots, n-2.$$

With regard to Problem 2) we remark that in $\overline{\Omega}_\varepsilon$ there exist «trivial» periodic bounce trajectories (i.e. $[A_\varepsilon, A_{-\varepsilon}]$ when $A_\varepsilon \in D_\varepsilon$, $A_{-\varepsilon} \in D_{-\varepsilon}$ and $p_0(A_\varepsilon) = p_0(A_{-\varepsilon})$). However we may state the existence of non-trivial periodic bounce trajectories in Ω_ε. Namely:

2. THEOREM. *For every $\eta > 0$, there exists a periodic bounce trajectory $\alpha = [z_1, \ldots, z_n]$ such that*

i) *if $z_i \in D_a$ then $z_{i+1} \in D_{-a}$* *for $i = 1, \ldots, n-1$*

ii) $\max\left(|\vartheta(z_i, z_{i+1})|, |\vartheta(z_{i+1}, z_i)|\right) < \eta$ *for $i = 1, \ldots, n-1$*

iii) $l_0(\alpha) = L_0$.

3. COROLLARY. *For every $\eta > 0$ there exists a periodic bounce trajectory α in Ω_ε such that $d(z, \alpha) < \eta$ for every $z \in \overline{\Omega}_\varepsilon$.*

Proofs will be given in successive papers.

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