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Meccanica dei solidi e delle strutture. — *Extremum theorem and convergence criterion for an iterative solution to the finite-step problem in elastoplasticity with mixed nonlinear hardening.* Nota di CLAUDIA COMI e GIULIO MAIER (*), presentata (**) dal Corrisp. G. MAIER.

ABSTRACT. — For a class of elastic-plastic constitutive laws with nonlinear kinematic and isotropic hardening, the problem of determining the response to a finite load step is formulated according to an implicit backward difference scheme (stepwise holonomic formulation), with reference to discrete structural models. This problem is shown to be amenable to a nonlinear mathematical programming problem and a criterion is derived which guarantees monotonic convergence of an iterative algorithm for the solution of the finite-step analysis problem. This communication anticipates in an abbreviated form results to be presented elsewhere in an extended form: here proofs and various comments are omitted.

KEY WORDS: Elastoplasticity; Finite-step; Extremum theorem; Convergence.

RIASSUNTO. — *Teorema di estremo e criterio di convergenza per una risoluzione iterativa del problema di passo finito in elastoplasticità con incrudimento misto non lineare.* Per una classe di leggi costituitive elasto-plastiche con incrudimento nonlineare cinematico ed isotropo, il problema relativo alla determinazione della risposta ad un passo di carico finito viene formulato in base ad uno schema implicito per «differenza all'indietro» (formulazione olonoma nel passo) con riferimento a modelli strutturali discreti. Il problema è ricondotto alla programmazione nonlineare e se ne deduce un criterio di convergenza monotona di un algoritmo iterativo per la risoluzione del problema di analisi nel passo finito. In questa nota alcuni risultati da presentare altrove in forma più estesa e dettagliata vengono comunicati in forma abbreviata omettendo dimostrazioni e vari commenti.

1. INTRODUCTION.

Boundary value problems in plasticity require to follow the whole evolution of a solid or structure as the external actions, assigned on the boundary Γ and in the domain Ω, vary in time t. In fact, the constitutive laws categorized as plastic are intrinsically path-dependent or «non holonomic» (and also homogeneous of order zero in time, so that in quasi static problem time t is a variable which merely orders events). Even when compatibility and equilibrium equations are linear (as we will assume here), the solution consists of integrating in space and time a system of nonlinear partial differential equations (supplemented by constitutive inequalities).

Approximate numerical solutions to engineering purposes rest on discretizations in space and time. The former discretization represents a fairly well understood and routinely performed procedure of computational mechanics and is of no concern in this paper, where a suitable standard finite element model will be assumed in order to simplify at most the formalism. The latter discretization (in time) is the subject of extensive current research and will be considered here with reference to a class of material models entailing nonlinear mixed hardening (kinematic and isotropic). The

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time integration is reduced to a sequence of nonlinear path-independent (holonomic) analyses for subsequent finite increments (steps) of the external actions, while the path-dependent, irreversible nature of plasticity is allowed for between two subsequent steps by an up-dating provision. The computation of the response to a finite step of external actions (loading step) is carried out by an iterative procedure consisting of a linear prediction phase and a suitable nonlinear correction phase, the latter being given by a «backward-difference» implicit formulation. Such an approximate time integration scheme is not new; in fact, since a few years it is becoming more and more popular in computational plasticity.

The contributions presented in this paper can be outlined as follows: (*a*) for the broad class of material models adopted, the incremental (nonholonomic, rate) constitutive law turns out to generate directly its stepwise holonomic counterpart, the use of which is equivalent to the above mentioned backward difference formulation of the finite-step problem; (*b*) under suitable constitutive restrictions, the solution to this problem is shown to be equivalent to the constrained minimization of a functional in the finite increments of kinematic variables: in other terms, the finite step problem is cast into a nonlinear programming problem; (*c*) under the same restrictions, using the above extremum theorem, a criterion is derived which guarantees monotonic convergence of the aforementioned iterative solution to the finite-step analysis problem.

The most pertinent earlier results related to the present contributions can be listed as follows.

In the Seventies piecewise linearization of yield surfaces was fairly frequently adopted in order to reduce elastic-plastic analysis to a sequence of quadratic programming problems, each of them defining the holonomic response of the system to a (finite) load step [3, 7]. Approximate time integration in quasi-static plasticity with nonlinear yield functions was extensively studied in recent years (see *e.g.* [12, 13, 16]), Martin and his coworkers vigorously dealt with this subject, pointing out the mechanical soundness of the implicit backward difference integration scheme and its links with the concepts of stepwise holonomic formulations and optimal paths, thus, in a sense, unifying the two above lines of thoughts. The main contributions of Martin and his group are quoted and summarized in [11].

Among the earlier noteworthy investigations in this area, we quote here the work done in the early Eighties by R. Casciaro and coworkers on convergence of iterative algorithms in plasticity: their main results are outlined in [1] with complete bibliography.

In both papers [1] and [11] just appeared a convergence criterion, similar to the one here presented in sect. 5, is independently expounded: in the former as a re-presentation of earlier results, in the latter as a novel result. However, in our opinion, this may not deprive of interest the present conclusions, since the context of multisurface (nonsmooth) plasticity with mixed nonlinear hardening here considered is more general and the approach and path of reasoning followed here are quite distinct from those followed by the quoted Authors.

As for extremum characterizations of step solutions, recent loosely related work concerns elastoplasticity without hardening [5] and with nonlinear isotropic hardening alone [8, 9].

Notation. Dots over symbols denote time derivatives. Matrices are indicated by bold-face symbols, transpose by a tilde. Vector inequalities apply componentwise. In order to remove the ambiguity which may arise because parentheses enclose both an argument and a factor, the latter case will be marked by a dot. Other symbols are defined where they are used for the first time.

2. SPACE DISCRETIZATION AND CONSTITUTIVE LAWS.

2.1. Consider a compatible finite element model of the body, encompassing m constant-strain finite elements run by index i. Let the s-vectors $\boldsymbol{q}^i$ and $\boldsymbol{Q}^i$ contain s «natural» generalized strains and stresses of element i. They are such that their scalar product $\widetilde{\boldsymbol{Q}}^i \delta \boldsymbol{q}^i$ represents virtual work performed on element i, and are not affected by rigid body motions.

In threedimensional problems element i is a four-nodes, 12 d.o.f. tetrahedron, $s=6$, $\boldsymbol{q}^i$ describe edge elongations, $\boldsymbol{Q}^i$ define self-equilibrated pairs of edge forces. If the element is homogeneous as for material properties, one-to-one linear equations relate the material strain components (gathered in the 6-vector $\boldsymbol{\varepsilon}^i$, with «engineering definition» of shear strains) to $\boldsymbol{q}^i$ and $\boldsymbol{Q}^i$ to the stress components (6-vector $\boldsymbol{\sigma}^i$, account taken of symmetry):

$$\boldsymbol{q}^i = \boldsymbol{T}^i \boldsymbol{\varepsilon}^i; \qquad \boldsymbol{\sigma}^i = \frac{1}{V^i} \widetilde{\boldsymbol{T}}^i \boldsymbol{Q}^i . \tag{2.1}$$

Where V^i denotes the element volume and $\boldsymbol{T}^i$ a (nonsingular) matrix depending only on the element geometry (vertex coordinates). The contragradient relations (2.1) makes it equivalent to deal with material behaviour in strain and stress tensors or with «element behaviour» in generalized variables, since one closely reflects the other. We shall adopt below the latter option for the sake of brevity and compactness of notation.

The class of «constitutive» laws we assume here for the local deformability is defined by the following relationship (for $i = 1 \dots m$):

$$\boldsymbol{Q}^i = \boldsymbol{E}^i \boldsymbol{e}^i; \qquad \boldsymbol{q}^i = \boldsymbol{e}^i + \boldsymbol{p}^i + \boldsymbol{\theta}^i \tag{2.2a,b}$$

$$\boldsymbol{\phi}^i \equiv \boldsymbol{\Phi}^i(\boldsymbol{\tau}^i) - \boldsymbol{Y}^i(\boldsymbol{\lambda}^i) \leqq \boldsymbol{0}; \qquad \text{where: } \boldsymbol{\tau}^i \equiv \boldsymbol{Q}^i - \boldsymbol{A}^i(\boldsymbol{p}^i) \tag{2.3a,b,c}$$

$$\dot{\boldsymbol{p}}^i = \boldsymbol{N}^i \dot{\boldsymbol{\lambda}}^i; \qquad \text{where: } \boldsymbol{N}^i \equiv \frac{\partial \widetilde{\boldsymbol{\psi}}^i}{\partial \boldsymbol{Q}^i}; \; \widetilde{\boldsymbol{\psi}}^i \equiv \{ \dots \psi^i_j(\boldsymbol{Q}^i, \boldsymbol{p}^i, \boldsymbol{\lambda}^i) \dots \} \tag{2.4a,b,c}$$

$$\dot{\boldsymbol{\lambda}}^i \geqq \boldsymbol{0}, \qquad \widetilde{\boldsymbol{\phi}}^i \dot{\boldsymbol{\lambda}}^i = 0 . \tag{2.5a,b}$$

The symbols employed can be specified as follows. Matrix $\boldsymbol{E}^i$ is the (symmetric, positive definite) element stiffness relating elastic (generalized) strains $\boldsymbol{e}^i$ to stresses. Eq. (2.2*b*) expresses the additivity of elastic, $\boldsymbol{e}^i$, plastic, $\boldsymbol{p}^i$, and imposed (*e.g.* thermal), $\boldsymbol{\theta}^i$, strains. Eqs. (2.3*a*) define the yield functions ϕ^i_j $(j = 1 \dots y)$ (collected in vector $\boldsymbol{\phi}^i$) as the sum of an effective stress Φ^i_j and of a yield limit Y^i_j (collected in the respective y-vectors). Each effective stress is seen to depend on the difference between stresses and a vector $\boldsymbol{A}^i$ depending in turn on plastic strains $\boldsymbol{p}^i$: the latter, generally nonlinear dependence (assumed as differentiable) is referred to as kinematic hardening, inasmuch it causes a rigid-body motion of the elastic domain defined by inequalities (2.3*b*). Each of the yield limits depends on y internal variables λ_j^i gathered in vector $\boldsymbol{\lambda}^i$: this (differentiable) dependence defines another kind of hardening called henceforth isotropic, since it implies a shape-preserving homotetic change of the yield surface in the special but frequent case of a single yield mode ($y = 1$) and of h order homogeneous effective stress ($\boldsymbol{\Phi}^i(\alpha\,\boldsymbol{\tau}^i) = \alpha^h\,\boldsymbol{\Phi}^i(\boldsymbol{\tau}^i)$ for any $\alpha \geqq 0$).

The internal variables are by eq. (2.5a) nondecreasing functions of time and nonnegative as well because we will assume $\boldsymbol{\lambda}^i = \mathbf{0}$ at $t = 0$. In eqs. (2.4), which formulate the generally nonassociative flow rule, $\boldsymbol{N}^i$ is a matrix whose columns can be regarded as stress gradients of plastic potentials ψ_j^i. If $\psi_j^i \equiv \Phi_j^i$ normality holds, *i.e.* the flow rule is associated. In view of eq. (2.4), each variable λ_j^i ($j = 1 \dots y$) can be said to represent a measure of the total amount of plastic flow (*i.e.* the contribution to plastic strains) due to the «activation» of the relevant yielding mode.

The complementarity eq. (2.5b) applies componentwise by virtue of the sign-constraints on the two vectors involved. Together with inequality (2.3b), it implies the complementarity relation

$$\dot{\tilde{\boldsymbol{\phi}}}^i\,\dot{\boldsymbol{\lambda}}^i = 0 \qquad (\text{or } \dot{\phi}_j^i\,\dot{\lambda}_j^i = 0 \text{ for } j = 1 \dots y) \tag{2.6}$$

which expresses Prager's «consistency» rule in rates.

2.2 Let us gather all elemental vectors (like $\boldsymbol{q}^i, \boldsymbol{\phi}^i$) and matrices (like $\boldsymbol{E}^i, \boldsymbol{N}^i$) affected by index i (for $i = 1 \dots m$) into corresponding entities indicated by the same symbols without index, *e.g.*:

$$\tilde{\boldsymbol{q}} \equiv \{\dots \tilde{\boldsymbol{q}}^i \dots\}, \qquad \tilde{\boldsymbol{\phi}} \equiv \{\dots \tilde{\boldsymbol{\phi}}^i \dots\}; \qquad \boldsymbol{E} \equiv \mathrm{diag}\,[\boldsymbol{E}^i], \qquad \boldsymbol{N} \equiv \mathrm{diag}\,[\boldsymbol{N}^i]. \tag{2.7}$$

Using these more comprehensive symbols the constitutive laws are rewritten for convenience in the form:

$$\boldsymbol{q} = \boldsymbol{E}^{-1}\boldsymbol{Q} + \boldsymbol{p} + \boldsymbol{\theta}(t) \tag{2.8}$$

$$\boldsymbol{\phi} = \boldsymbol{\Phi}(\boldsymbol{Q} - \boldsymbol{A}(\boldsymbol{p})) - \boldsymbol{Y}(\boldsymbol{\lambda}) \leqq \mathbf{0} \tag{2.9a,b}$$

$$\dot{\boldsymbol{p}} = \boldsymbol{N}\dot{\boldsymbol{\lambda}}, \qquad \dot{\boldsymbol{\lambda}} \geqq \mathbf{0}, \qquad \tilde{\boldsymbol{\phi}}\,\dot{\boldsymbol{\lambda}} = 0. \tag{2.10a,b,c}$$

The geometric compatibility and equilibrium equations read:

$$\boldsymbol{q} = \boldsymbol{C}\boldsymbol{u}, \qquad \tilde{\boldsymbol{C}}\boldsymbol{Q} = \boldsymbol{P}(t) \tag{2.11a,b}$$

where $\boldsymbol{u}$ denotes the vector of the (say n) free nodal displacements (degrees of freedom) and $\boldsymbol{P}$ the vector of the corresponding nodal loads equivalent (in the virtual work sense consistent with the displacement model) to the given body and surface loads.

For simplicity all kinematic boundary conditions are assumed homogeneous (given displacements can always be simulated by imposed strains $\boldsymbol{u}^i$ acting on suitably chosen stiff fititious elements). In eq. (2.11) matrix $\boldsymbol{C}$ is assumed to depend only on the undeformed geometry (small deformation hypothesis) and to have rank n (no kinematic indeterminacy).

The set of relations (2.8)-(2.11) fully governs the evolution of the elastic-plastic systems discretized in space under the given history of external actions represented by the vectors $\boldsymbol{\theta}(t)$ and $\boldsymbol{P}(t)$. We will assume $\boldsymbol{\theta}=\mathbf{0}$ in what follows, for the sake of brevity and simplicity.

3. Discretization in time and formulation of the step-holonomic problem.

Let us consider a sequence of «instants» $t_0=0,\ t_1 \dots t_{n-1},\ t_n=t_{n-1}+\Delta t_n, \dots$ of the ordinative «time» variable t along the evolution of the elastic-plastic system. These instants are chosen in such a way that at each of them the external actions (represented by the load vector $\boldsymbol{P}(t_n)\equiv \boldsymbol{P}_n$) are known and between two subsequent instants their variation can be assumed as proportional, so that the yielding processes they provoke can be reasonably expected to be regularly progressive over Δt_n or almost so (the expression «regularly progressive yielding» over Δt_n, introduced by Hodge, means that no yield mode which becomes active during the interval Δt_n undergoes unloading before its end t_n).

By (finite) step analysis we mean the determination of the finite increments of the response variables (*i.e.* of $\Delta \boldsymbol{u}_n$, $\Delta \mathrm{Q}_n$ etc.), when all variables are known in the starting situation at t_{n-1} and the load increment $\Delta \boldsymbol{P}_n$ is assigned.

The step analysis, by its very definition, formulates and solves algebraic relationships in the finite increments. It is meant to avoid and approximate the time integration of nonlinear differential relations of incremental path-dependent plasticity such as eqs. (2.8)-(2.11).

To simplify the notation, we drop henceforth all step indices and denote by Δ increments, by barred symbols the known quantities ($\overline{\boldsymbol{u}}$, $\overline{\mathrm{Q}}$ etc. at $t_{n-1}\equiv \bar{t}$; $\Delta\overline{\boldsymbol{P}}$ over $\Delta t_n \equiv \Delta t$; $\overline{\boldsymbol{P}}+\Delta\overline{\boldsymbol{P}}$ at instant $\bar{t}+\Delta t$), and by unbarred symbols the unknown quantities at t_n ($\boldsymbol{u}=\overline{\boldsymbol{u}}+\Delta\boldsymbol{u}$, etc).

Thus, adopting a backward difference implicit approach, we formulate as follows the above defined step analysis problem concerning the discrete (finite element) model described in sect. 2 and governed by eqs. (2.8)-(2.11):

$$\boldsymbol{\phi}=\boldsymbol{\Phi}(\boldsymbol{\tau})-\boldsymbol{Y}(\overline{\boldsymbol{\lambda}}+\Delta\boldsymbol{\lambda})\leqq \mathbf{0}; \qquad \text{where: } \boldsymbol{\tau}=\overline{\mathrm{Q}}+\Delta\mathrm{Q}-\boldsymbol{A}(\overline{\boldsymbol{p}}+\Delta\boldsymbol{p}) \tag{3.1a, b, c}$$

$$\Delta\boldsymbol{p}=\boldsymbol{N}\Delta\boldsymbol{\lambda}, \qquad \Delta\boldsymbol{\lambda}\geqq\mathbf{0}, \qquad \tilde{\boldsymbol{\phi}}\Delta\boldsymbol{\lambda}=0 \tag{3.2a, b, c}$$

$$(\Delta\boldsymbol{e}=)\,\boldsymbol{E}^{-1}\Delta\mathrm{Q}=\boldsymbol{C}\Delta\boldsymbol{u}-\boldsymbol{N}\Delta\boldsymbol{\lambda}, \qquad \widetilde{\boldsymbol{C}}\Delta\mathrm{Q}=\Delta\overline{\boldsymbol{P}}. \tag{3.3a, b, c}$$

Here matrix $\boldsymbol{N}$ is understood as evaluated for its (unknown) argument at the end (t_n) of the step Δt.

If Δt becomes infinitesimal ($\Delta t=\delta t$), then $\boldsymbol{N}$ becomes a known matrix, all increments are proportional to rates (*e.g.*: $\Delta\boldsymbol{\lambda}=\delta\boldsymbol{\lambda}=\dot{\boldsymbol{\lambda}}\,\delta t$), and eqs. (3.1) become linear by expansion of $\boldsymbol{\Phi}$, $\boldsymbol{A}$ and $\boldsymbol{Y}$ into Taylor series truncated at first order terms. Thus eqs. (3.1)-(3.3) reduce to the differential formulation of sect. 2, *i.e.* to a system consisting of linear equations and a linear complementarity problem in the rates. Such simple mathematical structure would be preserved in an explicit forward-difference (Euler) formulation of the finite step problem, but violations of the yield condition would occur (and would up along the step sequence if in the absence of suitable corrections). The implicit, backward difference formulation (3.1)-(3.3) instead includes linear equations and a nonlinear complementarity problem, but it avoids the above violations.

4. Restrictions on the constitutive laws and consequent extremum characterizations of the finite-step response.

4.1. We list and comment below the constitutive hypotheses under which the validity of the statements given later is subjected. Since these restrictions hold for all elements, the more comprehensive symbology (for all $i = 1 \dots m$, hence without index i) introduced at the end of sect. 2, will be used.

(*a*) Associated flow rules. The plastic potentials coincide with the yield functions:

$$(4.1a,b) \qquad \psi = \phi, \quad \text{so that:} \quad \frac{\partial \widetilde{\psi}}{\partial Q} = \frac{\partial \widetilde{\phi}}{\partial Q} = \frac{\partial \widetilde{\boldsymbol{\Phi}}}{\partial \tau}(\tau) = N(\tau), \qquad \text{where: } \tau = Q - A(p).$$

Thus one of the requirements of Drucker's stability postulate [4] is satisfied, namely the outward normality or «association» (of the plastic strain rate vector to the yield surface). It is worth noting to recall that materials with internal friction (e.q. concrete and geomaterials) are known to require nonassociative rules to be realistically described.

(*b*) Convexity of the yield functions in stresses. The effective stresses $\boldsymbol{\Phi}$ are convex functions of the argument τ; a necessary and sufficient condition for this is well known to be that their linear approximations around any point τ' represent lower bounds to their values, namely:

$$(4.2) \qquad \boldsymbol{\Phi}(\tau) \geqq \boldsymbol{\Phi}(\tau') + \widetilde{N}(\tau') \cdot (\tau - \tau'), \qquad \text{for any } \tau, \ \tau'.$$

As a consequence, the yield functions ϕ are convex functions of the stresses as well; hence, the yield inequalities (2.3), (2.9) and (3.1) define a convex elastic domain in each Q^i-space ($i = 1 \dots m$), according to another requirement of Drucker's postulate, which is fulfilled in most circumstances of engineering interest.

(*c*) Homogeneity of effective stresses. The effective stresses are positively homogeneous functions of order one of their argument, namely $\boldsymbol{\Phi}(\alpha\tau) = \alpha\boldsymbol{\Phi}(\tau)$ for any $\alpha \geqq 0$.

This implies, by Euler's theorem, that:

$$(4.3) \qquad \boldsymbol{\Phi}(\tau) = \frac{\partial \boldsymbol{\Phi}}{\partial \tilde{\tau}}(\tau)\, \tau = \widetilde{N}(\tau)\, \tau$$

and, consequently, that every vector τ^i satisfies the homogeneous equation associated to the Hessian, calculated in it, of each effective stress:

$$(4.4) \qquad \frac{\partial^2 \Phi^i_j}{\partial \tau^i \partial \tilde{\tau}^i}(\tau^i)\, \tau^i = 0 \qquad (j = 1 \dots y).$$

This hypothesis is fulfilled in the traditional and most widely used material models (e.g. Mises and Tresca and their various generalizations). It is not in even simple models of structural element behaviour (*e.g.*, beams in bending and stretching, idealized by the generalized plastic hinge notion, whose interaction curves are parabolic). However, even in such cases, the hypothesis of order-one homogeneity for the effective stresses can be shown to be satisfied without loss of generality by suitably reformulating the yield conditions.

(*d*) Reciprocal hardening. Consider the total plastic work namely the following function of time t and functional of the plastic strain history $\boldsymbol{p}(t')$, $0 \leqq t' \leqq t$:

$$W^p(t) \equiv \int_0^t \widetilde{\boldsymbol{Q}}(t')\,\dot{\boldsymbol{p}}(t')\,dt\,. \tag{4.5}$$

Using the flow rule (2.10*a*), the hypotheses of normality (*a*) and homogeneity (*c*) and the complementarity relation (2.10*c*), W^p splits into two addends, the former π associated to the so-called isotropic hardening, the latter Γ to kinematic hardening:

$$W^p(t) = \Pi(t) + \Gamma(t) \tag{4.6}$$

$$\Pi(t) \equiv \int_0^t \widetilde{\boldsymbol{Y}}(t')\,\dot{\boldsymbol{\lambda}}(t')\,dt'; \qquad \Gamma(t) \equiv \int_0^t \widetilde{\boldsymbol{A}}(t')\,\dot{\boldsymbol{p}}(t')\,dt'\,. \tag{4.7a, b}$$

We assume that these addends are functions of the internal variables and of the plastic strains at t, respectively, *i.e.* $\Pi(\boldsymbol{\lambda})$ and $\Gamma(\boldsymbol{p})$, and no longer functionals of their past histories. Necessary and sufficient conditions for these circumstances are, respectively:

$$\boldsymbol{H} \equiv \frac{\partial \widetilde{\boldsymbol{Y}}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \widetilde{\boldsymbol{H}}; \qquad \boldsymbol{K} \equiv \frac{\partial \widetilde{\boldsymbol{A}}}{\partial \boldsymbol{p}}(\boldsymbol{p}) = \widetilde{\boldsymbol{K}}\,. \tag{4.8a, b}$$

The symmetry expressed by (4.8) of the Jacobians, or «hardening matrices», of the two dependences (symmetry holding elementwise, for all i separately), can be mechanically interpreted as hardening «reciprocity».

Note that this assumption is not consequence of Drucker's or Iljushin's «quasi thermodynamical» postulates, nor of Hill's stability criterion. It is violated by some recently constitutive models proposed for geomaterials (*e.g.* Martin-Resende model [15]).

(*e*) Material stability. Both hardening matrices $\boldsymbol{H}$ and $\boldsymbol{K}$ are assumed to be positive semidefinite. This implies that the second-order variation of the plastic work cannot be negative, (neither in overall sense nor elementwise) because:

$$\delta^{(2)} W^p = \frac{1}{2}\,\delta\widetilde{\boldsymbol{Y}}(\delta\boldsymbol{\lambda})\,\delta\boldsymbol{\lambda} + \frac{1}{2}\,\delta\widetilde{\boldsymbol{A}}(\delta\boldsymbol{p})\,\delta\boldsymbol{p} = \frac{1}{2}\,\delta\widetilde{\boldsymbol{\lambda}}\,\widetilde{\boldsymbol{H}}\,\delta\boldsymbol{\lambda} + \frac{1}{2}\,\delta\widetilde{\boldsymbol{p}}\,\widetilde{\boldsymbol{K}}\,\delta\boldsymbol{p}\,. \tag{4.9}$$

Therefore, (second order) work can never be extracted from the material (or element or structural component) by an external agency which causes infinitesimal geometric changes by preserving equilibrium (so that first order work vanishes); thus softening behaviour is ruled out. This is Hill's sufficient criterion for stability and a further requirement of Drucher's stability postulate («in the small»).

Note that this hypothesis is necessary and sufficient for the convexity of both functions $\Pi(\boldsymbol{\lambda})$ and $\Gamma(\boldsymbol{p})$ by virtue of hypothesis (*d*).

4.2. On the basis of the above hypotheses the following statement can be proved.

PROPOSITION 1. Consider the (generally nonconvex) constrained optimization (nonlinear programming) problem.

$$\min_{\Delta \boldsymbol{u},\,\Delta \boldsymbol{e},\,\Delta \boldsymbol{\lambda}} \left\{ \Omega \equiv \frac{1}{2}\,\Delta\widetilde{\boldsymbol{e}}\,\boldsymbol{E}\,\Delta\boldsymbol{e} + \Pi(\overline{\boldsymbol{\lambda}} + \Delta\boldsymbol{\lambda}) + \Gamma(\overline{\boldsymbol{p}} + \Delta\boldsymbol{p}) - (\widetilde{\overline{\boldsymbol{P}}} + \Delta\widetilde{\overline{\boldsymbol{P}}})\,\Delta\boldsymbol{u} + \widetilde{\overline{\boldsymbol{Q}}}\,\Delta\boldsymbol{e} \right\} \tag{4.10}$$

subject to:

$$\Delta \boldsymbol{e} = \boldsymbol{C}\Delta \boldsymbol{u} - \Delta \boldsymbol{p}\,, \qquad \Delta\boldsymbol{\lambda} \geqq \mathbf{0}, \tag{4.11a, b}$$

$$\Delta \boldsymbol{p} = \boldsymbol{N}(\boldsymbol{E}\cdot(\overline{\boldsymbol{e}} + \Delta \boldsymbol{e}) - \boldsymbol{A}(\overline{\boldsymbol{p}} + \Delta \boldsymbol{p}))\,\Delta\boldsymbol{\lambda}\,. \tag{4.12}$$

Any solution to this problem solves the holonomic finite step problem (3.1)-(3.3) and also the converse holds true.

Note that the minimization process turns out to enforce both equilibrium and part of the constitutive law (precisely the complementarity, eq. (3.2*c*)), and the yield conditions, eq. (3.1*b*).

An extremum characterization in kinematic terms of the solution(*s*) to the analysis problem in finite increments formulated as path-independent within the step is expected, as a dual characterization is in static terms (in fact this will be presented and discussed in a parallel paper). However, the interest of the above theorem, in the writers' opinion, rests on the following circumstances: (*a*) the statement further generalizes earlier results [7], concerning piecewiselinear yield functions (and reducing the analysis to quadratic programming) and recent results such as those in [8, 9]; (*b*) its proof (expounded elsewhere) naturally leads to the above listed mechanically meaningful hypotheses which restrict the originally broad class of material models described in sect. 2; (*c*) extremum theorems of this kind privilege the backward difference approach to step-by-step time integration with respect to other heuristic approaches and witnesses its mechanical soundness; (*d*) the above theorem provides a foundation to a convergence proof of a computationally efficient solution algorithm, as shown in the next section.

5. CONVERGENCE CRITERION FOR BACKWARD-DIFFERENCE ITERATIVE ALGORITHMS.

5.1. The algorithm adopted here for the solution of the finite-step holonomic problem formulated in sect. 3 (or for the equivalent optimization problem according to Prop. 1) can be described as follows. We refer to the r-th iteration within the procedure applied to the n-th loading step over $\Delta t = t_n = t_n - t_{n-1}$: known quantities are the load increments $\Delta\overline{\boldsymbol{P}}$, all the variables (barred symbols) at the end of the preceding step Δt_{n-1} and the increments provided by the preceding iteration $r-1$ (specifically, the displacement increments, which are the only ones used). The current r-th iteration, as all the other ones, includes two phases: prediction and correction, plus, naturally, a termination test (comparing a norm of residuals to assigned tolerances).

A) *Prediction* phase: provides estimates of the displacement increments $\Delta\boldsymbol{u}^r$, by solving linearized equations for the whole (assembled) finite element model.

A.1) Compute residual loads $\boldsymbol{R}^r$ which measure the discrepancy between the given loads at the end of the step and the nodal forces required to equilibrate the stresses computed at the end of the preceding $(r-1)$ iteration:

$$\boldsymbol{R}^r \equiv \overline{\boldsymbol{P}} + \Delta\overline{\boldsymbol{P}} - \widetilde{\boldsymbol{C}}\cdot(\overline{\boldsymbol{Q}} + \Delta\boldsymbol{Q}^{r-1})\,. \tag{5.1}$$

A.2) Generate equations governing the fictitious linear-elastic response of the system to the residual (5.1), with a suitably chosen stiffness matrix $\boldsymbol{S}^r$:

$$\boldsymbol{S}^r\cdot(\Delta\boldsymbol{u}^r - \Delta\boldsymbol{u}^{r-1}) = \boldsymbol{R}^r\,. \tag{5.2}$$

A.3) Solve eq. (5.2), to obtain estimates $\Delta \boldsymbol{u}^r$ and, hence, $\boldsymbol{u}^r = \overline{\boldsymbol{u}} + \Delta \boldsymbol{u}^r$.

B) *Correction* phase: for the configuration changes $\Delta \boldsymbol{u}^r$ estimated by the predictor, computes through the (finite step holonomic) constitutive laws the increments of stress $\Delta \boldsymbol{Q}^r$. It involves solutions of nonlinear problems, but of small size as it operates at local, element level.

B.1) Compute the strain increments through the compatibility equation (2.11*a*):

$$q^r = \overline{q} + C \Delta u^r . \tag{5.3}$$

B.2) Solve with respect to $\Delta \boldsymbol{p}^r$, $\Delta \boldsymbol{\lambda}^r$ and $\boldsymbol{Q}^r$ the relation set, after having entered $\boldsymbol{q}^r$ as data:

$$\boldsymbol{Q}^r = \boldsymbol{E} \cdot (\boldsymbol{q}^r - \overline{\boldsymbol{p}}) - \boldsymbol{E} \Delta \boldsymbol{p}^r \tag{5.4}$$

$$\Delta \boldsymbol{p}^r = \boldsymbol{N}(\boldsymbol{Q}^r - \boldsymbol{A}(\overline{\boldsymbol{p}} + \Delta \boldsymbol{p}^r)) \Delta \boldsymbol{\lambda}^r ; \quad \Delta \boldsymbol{\lambda}^r \geqq \mathbf{0} \tag{5.5a, b}$$

$$\boldsymbol{\phi}^r = \boldsymbol{\Phi}(\boldsymbol{Q}^r - \boldsymbol{A}(\overline{\boldsymbol{p}} + \Delta \boldsymbol{p}^r)) - \boldsymbol{Y}(\overline{\boldsymbol{\lambda}} + \Delta \boldsymbol{\lambda}^r) \leqq \mathbf{0}, \quad \tilde{\boldsymbol{\phi}}^r \Delta \boldsymbol{\lambda}^r = 0 . \tag{5.6a, b, c}$$

B.3) Go to A.1 entering $\boldsymbol{Q}^r = \overline{\boldsymbol{Q}} + \Delta \boldsymbol{Q}^r$ into eq. (5.1).

5.2. In the above description of the algorithm the choice of the basis (*i.e.* element stiffnesses) on which the stiffness matrix $\boldsymbol{S}^r$ is computed was not specified. For such choice the following convergence theorem (whose proof is expounded in a parallel paper) provides a valuable criterion.

Proposition 2. Under the same hypotheses (*a*)-(*d*) of sect. 4 (those under which the extremum property of Prop. 1 is valid), if the element stiffnesses $\boldsymbol{E}^*$ employed for computing the prediction matrix $\boldsymbol{S}^r$ are assumed as:

$$\boldsymbol{E}^* = \boldsymbol{E} \quad \text{for } r = 1; \qquad \boldsymbol{E}^* = \frac{1}{\alpha} \boldsymbol{E} \quad \text{with } 0 < \alpha < 2 \text{ for } r > 1 \tag{5.7a, b}$$

($\boldsymbol{E}$ gathering as diagonal blocks the linear elastic stiffness of all elements), then the sequence of the values acquired at iteration r by function Ω definied by eq. (4.10) is monotonously decreasing:

$$\Omega(\Delta \boldsymbol{u}^{r-1}, \Delta \boldsymbol{p}^{r-1}, \Delta \boldsymbol{\lambda}^{r-1}) \geqq \Omega(\Delta \boldsymbol{u}^r, \Delta \boldsymbol{p}^r, \Delta \boldsymbol{\lambda}^r) \tag{5.8}$$

and the equality sign holds if and only if the solution to the finite step problem is attained. Therefore, when the above hypotheses are satisfied the procedure of sect. 5.1 does converge on the/a solution to the step problem.

It is worth noting that within the interval $\overline{02}$ postulated by condition (5.7*b*), the most favourable choice of α is generally the «least stiff» choice $\alpha \cong 2$, as for the speed of convergence is concerned.

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