A geometrically nonlinear analysis of laminated composite plates using a shear deformation theory

Giacinto Porco, Giuseppe Spadea, Raffaele Zinno


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Meccanica. — A geometrically nonlinear analysis of laminated composite plates using a shear deformation theory. Nota di GIACINTO PORCO (*), GIUSEPPE SPADEA (*) e RAFFAELE ZINNO (*), presentata (**) dal Socio E. GIANGRECO.

ABSTRACT. — A shear deformation theory is developed to analyse the geometrically nonlinear behaviour of layered composite plates under transverse loads.

The theory accounts for the transverse shear (as in the Reissner Mindlin plate theory) and large rotations (in the sense of the von Karman theory) suitable for simulating the behaviour of moderately thick plates.

Square and rectangular plates are considered: the numerical results are obtained by a finite element computational procedure and are given for various boundary and loading conditions, $a/b$ ratios, stacking and orientation of layers and material properties ($E_1/E_2$ ratio, $E_1/G$ ratio, etc.).

KEY WORDS: Plates; Composite; Laminated.

RIASSUNTO. — Sul comportamento non lineare di piastre laminate composite. In questo lavoro si sviluppa una teoria che tiene conto della deformabilità tagliante allo scopo di analizzare il comportamento di piastre laminate composite sottoposte a carichi flettenti.

La teoria tiene conto delle deformazioni dovute al taglio (nel senso della teoria delle piastre spesse di Reissner-Mindlin) e di rotazioni moderatamente grandi (nel senso della teoria di von Karman).

I risultati numerici, relativi a piastre rettangolari, sono stati ottenuti attraverso una procedura computazionale agli elementi finiti considerando varie condizioni di vincolo e di carico, diversi valori del rapporto $a/b$, differenti spessori ed orientazioni delle lamine e proprietà dei materiali (rapporto $E_1/E_2$, rapporto $E_1/G$, etc.).

NOTATION

\begin{align*}
A_{ij}, B_{ij}, D_{ij} & \quad \text{Extensional, flexural-extensional and flexural stiffness (i, j = 1, 2, 3)} \\
H_{ij} & \quad \text{Thickness shear stiffness (i, j = 4, 5)} \\
a, b & \quad \text{Plate dimensions along the X and Y directions, respectively} \\
E_1, E_2 & \quad \text{Layer elastic moduli in directions along fibres and normal to them, respectively} \\
G_{12}, G_{13}, G_{23} & \quad \text{Layer in-plane and thickness shear moduli} \\
b & \quad \text{Total thickness of the laminate} \\
t_k & \quad \text{Thickness of k-th layer} \\
K & \quad \text{Shear correction coefficient} \\
x, y, z & \quad \text{Position coordinates in a cartesian system} \\
M_i, N_i & \quad \text{Stress couples and stress resultants, respectively (i = x, y, xy)} \\
Q_i & \quad \text{Shear stress resultants (i = x, y)} \\
\theta_i & \quad \text{Orientation of the k-th layer} \\
\varphi_x, \varphi_y & \quad \text{Slopes in the xz and yz planes} \\
\chi, \chi_x, \chi_y & \quad \text{Curvature components} \\
\epsilon_x, \epsilon_y, \epsilon_{xy} & \quad \text{Strain components} \\
\epsilon_1, \epsilon_2, \epsilon_{12} & \quad \text{Middle surface strain components} \\
n & \quad \text{Number of layers} \\
q_0 & \quad \text{Uniform transverse load} \\
N & \quad \text{Number of global nodes of the mesh in the F.E.M. discretization}
\end{align*}

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In the last two decades laminated plates have become an important structural element widely applied in the aerospace, medical, automotive and electronics industries [16].

This is mainly due to their high stiffness-to-weight ratio, coupled with the flexibility in the selection of the laminate scheme that can be tailored to match the design requirements. A comparison of results obtained from the classical plate theory (CPT) with the exact elasticity solutions indicates the necessity of considering transverse shear deformation in the analysis of layered composite plates [4].

In fact transverse shear effects are more pronounced in composite laminated plates than in isotropic plates because of their low transverse shear moduli relative to the in-plane Young’s moduli (25 ÷ 40 instead of 2.6, typical of isotropic materials).

Unfortunately, much of the previous research in this field is limited to linear problems and the main results are based on the classical thin-plate theory, which neglects transverse shear deformation effects [3].

The work associated with anisotropic plates with an arbitrary number of layers reached a peak in 1969.

Whitney and Leissa [17] formulated the governing equations of generally laminated anisotropic plates analogous to the von Karman plate equation including stretching-bending coupling and in-plane rotatory inertia coupling. Plates considered were cross-ply (0°/90°/0°/90°...) or angle-ply (θi−θi−θi−θi) laminates.

Solutions to the problems of displacements due to transverse and lateral static loadings were presented using trigonometric functions after linearising the general von Karman equations.

The most extensive work on plates of composite materials in one volume is the book by Ashton and Whitney [1] published in 1970.

In 1967 Pagano [6-7-18] included transverse shear deformation in the analysis of a bidirectional composite beam. Pagano published two papers in 1970 [8-9]: one dealing with cylindrical bending and the other with rectangular bidirectional composite layered plates.

By using the F.S.D.T. (First Shear Deformation Theory) [12-13], because of different Young’s moduli in two adjacent layers, there is a discontinuity of normal stresses at the interface between sheets.

The aim of the present work deals with the nonlinear analysis of laminated composite plates in the case of transverse loadings. The analysis is developed by using a finite element technique based on a shear deformation theory which accounts for the transverse shear also including nonlinear terms in the strain-displacement relationship in the sense of von Karman.

Numerical results are presented giving deflections of square and rectangular plates with various edge boundary conditions.
Governing equations of moderately thick plate model.

The type of plate under discussion consists of \( n \) layers of orthotropic sheets bonded together.

The origin of the coordinate system \((x, y)\) is taken in the middle plane \( \Omega \) of the plate with the \( z \) axis perpendicular to it. Each layer has an arbitrary thickness; the elastic properties and the orientation of orthotropic axes of each layer, with respect to the plate axes, are also arbitrary.

It is well-known, from experimental observations, that the Kirchhoff-Love theory of thin plates (in which it is assumed that normals to the midplane before deformation remain straight and normal to the plane after deformation) underpredicts deflections.

These results derive from neglect of transverse shear strains in the classical plate theory. For plates with side-to-thickness ratios greater than 20 the transverse normal and shear stresses are negligible when compared to the remaining stresses.

The inaccuracy of these previous results is faced in this work by using the Mindlin-Reissner plate theory [5-14] in which plane sections originally perpendicular to the middle plane of the plate remain plane, but not necessarily perpendicular to it.

Then, the displacement field is given by:

\[
\begin{align*}
\begin{bmatrix}
u(x, y, z) & = & u_0(x, y) + z\psi_x(x, y) \\
v(x, y, z) & = & v_0(x, y) + z\psi_y(x, y) \\
w(x, y, z) & = & w_0(x, y)
\end{bmatrix}
\tag{1}
\end{align*}
\]

in which \( u, v, w \) are, respectively, the displacements in the \( x, y, z \) directions, \( u_0, v_0, w_0 \) are the corresponding midplane displacements and \( \psi_x, \psi_y \) are the bending slopes in the \( xz \) and \( yz \) planes.

Assuming that the transverse deflection is comparable to the total thickness of the plate and strains are much smaller than rotations, the nonlinear strain-displacement

\[\text{Fig. 1. - Plate model.}\]
relations can be taken as:

\[
\begin{aligned}
\varepsilon_{xx} &= u_{0,x} + \frac{1}{2} w_{0,y}^2 + z\psi_{x,x} \\ 
\varepsilon_{yy} &= v_{0,y} + \frac{1}{2} w_{0,y}^2 + z\psi_{y,y} \\ 
\varepsilon_{zz} &= \frac{1}{2} (\psi_x^2 + \psi_y^2) \\ 
2\varepsilon_{zx} &= \psi_x + w_x \\ 
2\varepsilon_{zy} &= \psi_y + w_y \\ 
2\varepsilon_{xy} &= u_{0,y} + v_{0,x} + w_{0,x} w_{0,y} + z(\psi_{x,y} + \psi_{y,x})
\end{aligned}
\]

(2)

Given that:

\[
\begin{aligned}
\varepsilon_x^0 &= u_{0,x} + \frac{1}{2} w_{0,x}^2 \\
\varepsilon_y^0 &= u_{0,y} + \frac{1}{2} w_{0,y}^2 \\
\gamma_x &= \psi_{x,x} \\
\gamma_y &= \psi_{y,y} \\
\gamma_{xy} &= \psi_{x,y} + \psi_{y,x} \\
\gamma_{yz} &= \psi_{y,x} + \psi_{x,y}
\end{aligned}
\]

(3)

Then:

\[
\begin{aligned}
\varepsilon_{xx} &= \varepsilon_x^0 + z\chi_x \\
\varepsilon_{yy} &= \varepsilon_y^0 + z\chi_y \\
2\varepsilon_{xy} &= \varepsilon_{xy} + z\chi_{xy} \\
2\varepsilon_{zz} &= \gamma_{xx} \\
2\varepsilon_{xz} &= \gamma_{yx}
\end{aligned}
\]

(4)

Fig. 2. – Effect of shear deformation (β: slope due to the shear deformation).
Furthermore, $\varepsilon_{xx}$ is neglected since the constitutive relations are based on the plane-stress assumption.

With reference to fig. 3, the constitutive equations of the $k$-th layer are:

$$
\begin{pmatrix}
\sigma_{11}^k \\
\sigma_{22}^k \\
\sigma_{12}^k \\
\sigma_{23}^k
\end{pmatrix} =
\begin{bmatrix}
C_{11}^k & C_{12}^k & 0 & 0 \\
C_{21}^k & C_{22}^k & 0 & 0 \\
0 & 0 & C_{33}^k & 0 \\
0 & 0 & 0 & C_{44}^k
\end{bmatrix}
\begin{pmatrix}
\varepsilon_{11}^k \\
\varepsilon_{22}^k \\
\varepsilon_{12}^k \\
\varepsilon_{23}^k
\end{pmatrix}
$$

(5)

where $\sigma_{ij}^k$ and $\varepsilon_{ij}^k$ are the components of stress and strain tensors, respectively, defined in the material coordinates, and $\mathbf{C}_0^k$ are the material stiffness coefficients.

The coefficients $\mathbf{C}_0^k = \mathbf{C}_0^k$ are given (in terms of engineering constants) by:

$$
\begin{align*}
C_{11}^k &= E_1^k / (1 - \nu_{12}^k \nu_{21}^k) \\
C_{12}^k &= \nu_{12} E_2^k / (1 - \nu_{12}^k \nu_{21}^k) \\
C_{21}^k &= \nu_{21} E_1^k / (1 - \nu_{12}^k \nu_{21}^k) \\
C_{22}^k &= E_2^k / (1 - \nu_{12}^k \nu_{21}^k) \\
C_{33}^k &= G_{12}^k \\
C_{44}^k &= G_{13}^k \\
C_{55}^k &= G_{23}^k
\end{align*}
$$

(6)

where $G_{12}^k$, $G_{13}^k$, $G_{23}^k$ are layer in-plane and thickness shear moduli and $E_1^k$, $E_2^k$ are, respectively, the layer elastic moduli in directions along fibres (direction 1) and normal to them (direction 2).

As a consequence of the symmetry conditions:

$$
\nu_{12}^k / E_1^k = \nu_{21}^k / E_2^k
$$

(7)

for which:

$$
\nu_{21}^k = \frac{E_2^k}{E_1^k} \nu_{12}^k
$$

(8)
Then, for an orthotropic elastic medium there are only six independent elastic constants, when the constitutive relations are based on the plane-stress assumption.

For \( k \)-th layer, the stress-strain relation, with respect to the plate axes \( x, y, z \), is obtained from:

\[
\{\sigma^k\} = [T^k] [\varepsilon^k] [T^k]^{-1} \{\varepsilon^k\} = [C^k] \{\varepsilon^k\}
\]

where \( \{\sigma^k\} \) and \( \{\varepsilon^k\} \) are the components of stress and strain tensors defined with reference to the plate axes.

\[
\begin{align*}
\{\sigma^k\} &= \begin{bmatrix} \sigma_{xx}^k \\ \sigma_{yy}^k \\ \tau_{xy}^k \\ \tau_{yx}^k \\ \tau_{zx}^k \\ \tau_{xz}^k \end{bmatrix} ; \\
\{\varepsilon^k\} &= \begin{bmatrix} \varepsilon_x^k \\ \varepsilon_y^k \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yx} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{zx} \end{bmatrix}
\end{align*}
\]

The transformation matrix \( T^k \) is given by:

\[
[T^k] = \begin{bmatrix} A^k & B^k & 2C^k & 0 & 0 \\ B^k & A^k & -2C^k & 0 & 0 \\ -C^k & C^k & D^k & 0 & 0 \\ 0 & 0 & 0 & m^k & n^k \\ 0 & 0 & 0 & -n^k & m^k \end{bmatrix}
\]

in which:

\[
\begin{align*}
m^k &= \cos \theta_k \\
n^k &= \sin \theta_k \\
A^k &= \cos^2 \theta_k \\
B^k &= \cos^2 \theta_k - \sin^2 \theta_k \\
C^k &= \sin \theta_k \cos \theta_k \\
D^k &= \cos^2 \theta_k - \sin^2 \theta_k
\end{align*}
\]

The coefficients \( C^k_{ij} \) are given by:

\[
\begin{align*}
C_{11} &= C_{11}^k \cos^4 \theta_k + 2(C_{12} + 2C_{13}) \cos^2 \theta_k \sin^2 \theta_k + C_{22} \sin^4 \theta_k \\
C_{12} &= (C_{11} + C_{22} - 4C_{13}) \sin^2 \theta_k \cos^2 \theta_k + C_{12} \sin 2\theta_k \cos 2\theta_k \\
C_{13} &= (-C_{11} + C_{22} + 2C_{13}) \sin \theta_k \cos \theta_k + (-C_{12} + C_{22} - 2C_{13}) \sin^3 \theta_k \cos \theta_k \\
C_{22} &= C_{11} \sin^2 \theta_k + 2(C_{12} + 2C_{13}) \sin^2 \theta_k \cos^2 \theta_k + C_{22} \cos^4 \theta_k \\
C_{31} &= C_{13} \\
C_{32} &= C_{12} \\
C_{33} &= C_{23} \\
C_{34} &= C_{34} = C_{43} \\
C_{35} &= (C_{11} + C_{22} - 2C_{13}) \sin^2 \theta_k \cos^2 \theta_k + C_{33} \sin 2\theta_k \cos 2\theta_k \\
C_{36} &= (-C_{11} + C_{22} + 2C_{13}) \sin^3 \theta_k \cos \theta_k + (-C_{12} + C_{22} - 2C_{13}) \sin \theta_k \cos^3 \theta_k \\
C_{42} &= C_{13} \cos^2 \theta_k + C_{23} \sin^2 \theta_k \\
C_{52} &= (C_{23} - C_{33}) \sin \theta_k \cos \theta_k \\
C_{53} &= C_{53} \sin^2 \theta_k + C_{23} \cos^2 \theta_k
\end{align*}
\]

The total potential energy of the plate, in absence of body forces and neglecting both body moments and surface shearing forces, is given by:

\[
\Pi = U + V
\]
where:

\( U = \frac{1}{2} \int_{\Omega} \sigma \varepsilon \, dV = \frac{1}{2} \int_{\Omega} \sum_{k=1}^{n} \int_{b_k}^{h_k} (\sigma_{xx}^k \varepsilon_{xx}^k + \sigma_{yy}^k \varepsilon_{yy}^k + 2\tau_{xy}^k \varepsilon_{xy}^k + 2\tau_{zx}^k \varepsilon_{zx}^k + 2\tau_{zy}^k \varepsilon_{zy}^k) \, dz \, d\Omega \)  

(16) \( V = -\int_{\Omega} q_0 \psi_0 \, d\Omega - \sum_{C_1}^{C_4} \int_{N_u} N_u \psi_u \, dC_1 - \int_{C_2}^{C_3} \int_{N_i} N_i \psi_i \, dC_2 + \) 

\[ \int_{C_4}^{C_5} \int_{M_x} M_x \psi_x \, dC_3 - \int_{C_4}^{C_5} \int_{M_y} M_y \psi_y \, dC_4 - \int_{C_4}^{C_5} \int_{Q_x} Q_x \psi_x \, dC_5 \]

\( C_1, C_2, C_3, C_4, C_5 \) are, respectively, the (possibly overlapping) portions of the boundary on which \( N_u, N_i, M_x, M_y, Q_x \) are specified.

It is helpful to define the stress and moment resultants as follows:

\[
\begin{align*}
N_x &= \sum_{k=1}^{n} \int_{b_k}^{h_k} \sigma_{xx}^k \, dz; & N_y &= \sum_{k=1}^{n} \int_{b_k}^{h_k} \sigma_{yy}^k \, dz; \\
M_x &= \sum_{k=1}^{n} \int_{b_k}^{h_k} \sigma_{zx}^k \, dz; & M_y &= \sum_{k=1}^{n} \int_{b_k}^{h_k} \sigma_{zy}^k \, dz; \\
N_{xy} &= N_{yx} = \sum_{k=1}^{n} \int_{b_k}^{h_k} \tau_{xy}^k \, dz; & M_{xy} &= M_{yx} = \sum_{k=1}^{n} \int_{b_k}^{h_k} \tau_{xy}^k \, dz; \\
Q_x &= \sum_{k=1}^{n} \int_{b_k}^{h_k} \tau_{zx}^k \, dz; & Q_y &= \sum_{k=1}^{n} \int_{b_k}^{h_k} \tau_{zy}^k \, dz. 
\end{align*}
\]  

(17)

The symbols \( h_k \) and \( h_{k+1} \) denote, respectively, the distances from the midplane to the lower and upper surface of the \( k \)-th layer.

The solution to the problem can be reached by using a variational form of the total potential energy functional.

Therefore, by taking into account the relations (17) and (2, 3, 4), the stationary condition to this functional gives:

\[
\delta \Pi = \int_{\Omega} \left( (N_x \delta \varepsilon_{xx}^k + N_y \delta \varepsilon_{yy}^k + N_{xy} \delta \varepsilon_{xy}^k + M_x \delta \chi_x + M_y \delta \chi_y + M_{xy} \delta \chi_{xy} + \\
+ Q_x \delta \chi_x + Q_y \delta \chi_y - q_0 \delta w) \right) \, d\Omega - \sum_{C_1}^{C_4} \int_{N_u} N_u \delta u \, dC_1 - \int_{C_2}^{C_3} \int_{N_i} N_i \delta i \, dC_2 + \\
- \int_{C_4}^{C_5} \int_{M_x} M_x \delta x \, dC_3 - \int_{C_4}^{C_5} \int_{M_y} M_y \delta y \, dC_4 - \int_{C_4}^{C_5} \int_{Q_x} Q_x \delta w \, dC_5 = 0.
\]  

(18)

On the other hand, eq. (18), by using the virtual displacements principle, becomes:

\[
\delta \Pi = \int_{\Omega} \left( (\partial u + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - q_0) \delta w + \left( \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u_x + \left( \frac{\partial M_x}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y \right) \delta u_y \right) \, d\Omega = 0
\]  

(19)
where:

$$\varepsilon = \frac{\partial}{\partial x} \left( N_x \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left( N_y \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial x} \left( N_{xy} \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial y} \right).$$

Now, the following matrices can be defined:

$$A_{ij} = \sum_{k=1}^{n} \int_{b_k}^{b_{k+1}} C_{ij}^k dz = \sum_{k=1}^{n} C_{ij}^k (b_k - b_{k-1}) \quad (i, j = 1, 2, 3) \quad \text{(Extensional stiffness);}$$

$$B_{ij} = \sum_{k=1}^{n} \int_{b_k}^{b_{k+1}} C_{ij}^k \frac{\partial^2 w}{\partial x^2} dz = \frac{1}{2} \sum_{k=1}^{n} C_{ij}^k (b_k^2 - b_{k-1}^2) \quad (i, j = 1, 2, 3) \quad \text{(Flexural-extensional stiffness);}$$

$$D_{ij} = \sum_{k=1}^{n} \int_{b_k}^{b_{k+1}} C_{ij}^k \frac{\partial^2 w}{\partial y^2} dz = \frac{1}{2} \sum_{k=1}^{n} C_{ij}^k (b_k^2 - b_{k-1}^2) \quad (i, j = 1, 2, 3) \quad \text{(Flexural stiffness);}$$

$$H_{ij} = K^2 \sum_{k=1}^{n} \int_{b_k}^{b_{k+1}} C_{ij}^k \frac{\partial^2 w}{\partial x \partial y} dz = K^2 \sum_{k=1}^{n} C_{ij}^k (b_k - b_{k-1}) \quad (i, j = 4, 5) \quad \text{(Thickness shear stiffness).}$$

where $K^2 = 5/6$ is the square of the shear correction coefficient [5].

Combining eq. (20) with relations (9), (10), (17) and (4) the following plate constitutive equations are obtained:

(21) $$\{N\} = [A] \{e_0\} + [B] \{\chi\}$$

(22) $$\{M\} = [B] \{e_0\} + [D] \{\chi\}$$

(23) $$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \begin{bmatrix} H_{44} & H_{45} \\ H_{54} & H_{55} \end{bmatrix} \begin{bmatrix} \gamma_{xx} \\ \gamma_{yy} \end{bmatrix}$$

where:

(24) $$\{N\} = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}$$

(25) $$\{e_0\} = \begin{bmatrix} e_{x0} \\ e_{y0} \\ e_{sy0} \end{bmatrix}$$

(26) $$\{\chi\} = \begin{bmatrix} \chi_x \\ \chi_y \end{bmatrix}$$

Finite element formulation.

In this section a finite element model based on the variational formulation of the equation (18) is presented.
The region $\Omega$ is divided into a finite number of isoparametric rectangular elements [10, 11].

Over each element the generalized displacements $(u, v, w, \psi_x, \psi_y)$ are interpolated by:

\[
\begin{align*}
    u &= \sum_{i=1}^{N} u_i f_i; \\
    v &= \sum_{i=1}^{N} v_i f_i; \\
    w &= \sum_{i=1}^{N} w_i f_i; \\
    \psi_x &= \sum_{i=1}^{N} \psi_{xi} f_i; \\
    \psi_y &= \sum_{i=1}^{N} \psi_{yi} f_i;
\end{align*}
\]

where $u_i$, $v_i$, $w_i$, $\psi_{xi}$, $\psi_{yi}$ are the values of the unknown functions at the global nodes of the mesh and $f_i$ are the interpolation functions.

The substitution of relations (26) into eq. (18) gives:

\[
Ku = F
\]

where $U$ collects the nodal values of the generalized displacements $u, v, w, \psi_x, \psi_y$; $K$ is the stiffness matrix of the plate, and $F$ is the nodal force vector.

It should be observed that the stiffness matrix $K$ depends on the solution $U$. Therefore, a standard iterative procedure must be used.

The coefficients of the stiffness matrix $K$ (secant) are given in the appendix.

**Numerical results and concluding remarks.**

In this section the formulations obtained above are used in the nonlinear analysis of square and rectangular moderately thick plates.

Tables I and II contain, respectively, a list of the materials and the boundary condition considered here.

Plates are subject to a uniformly distributed load applied on the top surface and acting in direction $z$.

In all computations a mesh of $(3 \times 3)$ nine-node quadratic elements is used.

It should be observed that no noticeable effect of the integration of shear terms is found in calculations for the quadratic elements used.

Fig. 4 shows the ratio between the centre deflection obtained using the shear deformation theory and that obtained using the classical thin plate theory vs. the side-to-thickness ratio.

This is given for an isotropic plate, two layer and eight layer cross-ply square laminates, subject to a uniform load.

The plate boundary conditions here considered are SS1 (simply supported).

**Table I.** - Mechanical properties of laminae.

<table>
<thead>
<tr>
<th>MATERIAL</th>
<th>$E_1/E_2$</th>
<th>$G_{12}/E_2$</th>
<th>$G_{13}/E_2$</th>
<th>$G_{23}/E_2$</th>
<th>$\nu_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BORON EPOXY</td>
<td>10.0</td>
<td>0.333</td>
<td>0.333</td>
<td>0.333</td>
<td>0.22</td>
</tr>
<tr>
<td>GRAPHITE EPOXY</td>
<td>40.0</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>GLASS EPOXY</td>
<td>3.0</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>MATERIAL I</td>
<td>25.0</td>
<td>0.5</td>
<td>0.5</td>
<td>0.2</td>
<td>0.25</td>
</tr>
</tbody>
</table>
The orthotropic layers are made of material I; the isotropic plate has: $E = E_2$ and $G = E/(2(1 + \nu_{12}))$.

The influence of shear deformability on bending is extremely relevant when $a/h \leq 20$.

The results above reaffirm the importance of including shear deformation in laminated composite structures.

Fig. 5 plots the nondimensional transverse centre deflection $W_c/h$ vs. the analysis of nondimensional transverse load $\tilde{q} = (q_0/E_2)(a/h)^4$ of a simply supported (SS1) square cross-ply plate [2].

Again, the layers are made of material I.

It should be observed that deflections are overestimated if a linear analysis is carried

\[ W = \frac{w E_2 h^3}{q_0 a^4} \]

<table>
<thead>
<tr>
<th>SIDE A</th>
<th>SIDE B</th>
<th>SIDE C</th>
<th>SIDE D</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS1</td>
<td>$u, w, \psi_x = 0$</td>
<td>$v, w, \psi_y = 0$</td>
<td>$u, w, \psi_x = 0$</td>
</tr>
<tr>
<td>CC2</td>
<td>$u, v, w, \psi_x = 0$</td>
<td>$u, v, w, \psi_y = 0$</td>
<td>$u, v, w, \psi_x = 0$</td>
</tr>
<tr>
<td>CC1</td>
<td>all edges clamped</td>
<td>$u, v, w, \psi_x, \psi_y = 0$</td>
<td></td>
</tr>
<tr>
<td>SC</td>
<td>$u, w, \psi_x = 0$</td>
<td>$u, v, w, \psi_x, \psi_y = 0$</td>
<td>$u, w, \psi_x = 0$</td>
</tr>
<tr>
<td>CSCS</td>
<td>$w, \psi_y = 0$</td>
<td>$v, w, \psi_y = 0$</td>
<td>$w, \psi_y = 0$</td>
</tr>
</tbody>
</table>

**Fig. 4.** - Effect of shear deformability.
out; however the difference between the linear and nonlinear solution decreases with layering.

A cursory examination of (20) reveals that the coupling between bending and stretching as displayed in eqs. (21) (22) vanishes as the number of layers increases.

Figs. 6a, b, c show the nondimensional centre deflection of $n$-layer square plates made of Graphite-Expoxy material vs. angle $\theta$ for various boundary conditions [15].

When the number of layers is high ($n \geq 6 \div 7$) the «uncoupled» solution is reached.

A vast difference can be observed between results relative to the linear and nonlinear analysis, especially when coupled behaviour takes place.

For the SS1 boundary conditions (fig. 6a), the linear analysis, in the case of two layers, shows rigidity at a maximum when $\theta = 0^\circ$ and $\theta = 45^\circ$ and a minimum when $\theta = 12.5^\circ$.

On the other hand the nonlinear analysis gives a single optimal value of $\theta = 45^\circ$.

However, if there are 4 or more layers (even up to and beyond 100) the linear solution is similar to the nonlinear one (with the minimum rigidity when $\theta = 0^\circ$, and the maximum when $\theta = 45^\circ$).

In fig. 6c, relative to the clamped scheme, it can be noted that, in the linear analysis, a plate manufactured with 2 layers has the maximum rigidity when $\theta = 0^\circ$.

For 4 layers and more the angle of minimum deflection is $\theta = 45^\circ$.

The nonlinear analysis shows a single optimal $\theta = 0^\circ$ for all types of plates.

The same behaviour seen in fig. 6c can be also be observed in fig. 6b which represents the SC boundary conditions.

Fig. 7 plots the nondimensional transverse centre deflection $W_c/h$ versus the $E_i/E_2$ ratio of a simply supported (SS1) cross-ply square plate.

In these calculi $G_{12} = G_{13} = G_{23} = 0.5E_2$; and $\nu_{12} = 0.25$ were assumed.

---

![Fig. 5. - Transverse centre deflection $W_c/h$ vs load $\bar{q} = (q_0/E_2)(a/b)^4$ influence of layering on deformability. ($L =$ linear, $NL =$ nonlinear).](image-url)
Fig. 6a, b, c. – Effect of fibre orientation on centre deflection of angle-ply square plates under various boundary conditions.

In particular the diagrams show the dependence of the coupling effect on the ratio $E_1/E_2$.

A sufficiently high transverse nondimensional load ($\bar{q} = (q_0/E_2)(a/h)^4 = 50$) was assumed, so that nonlinear behaviour would have an appreciable effect.

The influence of shear deformability on nonlinear bending is shown, in the same fig. 7, by the comparison between results relative to both ratios $a/h = 100$ and $a/h = 10$.

In particular the classical plate theory (simulated by $a/h = 100$) underpredicts deflections even at lower ratios of moduli.

Fig. 8 shows the variation of the nondimensional centre deflection vs. the $a/b$ ratio. Plates are in Graphite-Epoxy material. It should be observed that the influence of coupling may be extremely relevant; nevertheless, the centre deflection rapidly reaches the uncoupled solution as the number of layers increases.
Fig. 7. - Effect of material anisotropy on the nondimensional centre deflections of cross-ply laminates under a uniform transverse load for different $a/h$ ratios.

Fig. 8. - Centre deflections vs $a/b$ ratio and $a/h$ ratio for a square cross-ply plate under a uniform transverse load.

Fig. 9. - Centre deflections as a function of $E_I/G$ ratio for square cross-ply plate under a uniform transverse load.
Furthermore, when the $a/b$ ratio increases, the $W_c/h$ ratio becomes asymptotic, with different values for each type of laminate.

Fig. 9 shows the nondimensional centre deflections $vs.$ the $E_1/G$ ratio of a square cross-ply plate under SS1 boundary conditions.

The material considered has the following mechanical properties:

$$E_1/E_2 = 40; \quad \nu_{12} = 0.25; \quad G = G_{12} = G_{13} = G_{23}.$$  

An almost linear increase of deflections is noted with the increase of the $E_1/G$ ratio.

The results relative to the plate manufactured with two layers show a large difference between the results of both linear and nonlinear analysis when $a/h = 10$ and $a/h = 100$.

This difference is lower for a 4 layer plate which has an uncoupled behaviour.
Figs. 10a, b, c show the variation of the centre deflection, $W_c$, versus the fibre orientation of an angle-ply square composite plate made with various materials.

Two values of the $a/b$ ratio are considered; furthermore only the simply supported boundary conditions is analysed.

A clear difference between linear and nonlinear analysis can be observed, especially for the plate made of Glass-Epoxy.

In particular, for all materials (except Graphite-Epoxy in the linear analysis with $a/b = 100$), the linear and nonlinear solutions give an optimal value of $\theta = 45^\circ$.

In figs. 11a, b the effect of the thickness ratio $t_1/t_2$ on the deformability of a 3-layer square plate with an orthotropic core, under the SS1 and CC1 boundary conditions, is shown for different values of load $q$.

The orthotropic core is made of material I; for the isotropic layers we assumed:

$$E = E_1; \quad G = E/(2(1 + \nu_{12})); \quad \nu_{12} = 0.25.$$  

It can be observed that the linear and nonlinear deflections decrease with $t_1/t_2$.

Finally figs. 12a, b show the nondimensional centre deflection $\nu$ vs. the $b/a$ ratio for CC1 boundary conditions and for different values of load $q = (q_0/E_2)(a/b)^4$.

In particular in the case of fig. 12a the orthotropic core has the fibre direction perpendicular to the shorter side (modulus $E_1$).

This plate has a higher rigidity than the analogous plate with the core fibre orientation parallel to the major side.

A greater difference between linear and nonlinear solutions can be observed in fig. 12b.

In any case this difference is more pronounced when the $b/a$ ratio increases.

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Fig. 12a, b. 3-layer sandwich plate: influence of the aspect ratio on the maximum deflection $W_c/h$.

APPENDIX

**Coefficients of the secant stiffness matrix $K$**

\[
\begin{bmatrix}
K_{11} + K_{11}^L & K_{12} + K_{12}^L & K_{13} + K_{13}^L & K_{14} + K_{14}^L & K_{15} + K_{15}^L \\
K_{21} + K_{21}^L & K_{22} + K_{22}^L & K_{23} + K_{23}^L & K_{24} + K_{24}^L & K_{25} + K_{25}^L \\
K_{31} + K_{31}^L & K_{32} + K_{32}^L & K_{33} + K_{33}^L & K_{34} + K_{34}^L & K_{35} + K_{35}^L \\
K_{41} + K_{41}^L & K_{42} + K_{42}^L & K_{43} + K_{43}^L & K_{44} + K_{44}^L & K_{45} + K_{45}^L \\
K_{51} + K_{51}^L & K_{52} + K_{52}^L & K_{53} + K_{53}^L & K_{54} + K_{54}^L & K_{55}^L
\end{bmatrix}
\]

Linear coefficients

\[
K_{11} = \int (H_{44} \psi_x \delta w_x + H_{45} \psi_y \delta w_x + H_{54} \psi_x \delta w_y + H_{55} \psi_y \delta w_y) \, dx \, dy
\]

\[
K_{12} = \int (H_{45} \psi_x \delta w_x + H_{55} \psi_y \delta w_x) \, dx \, dy
\]

\[
K_{13} = \int (H_{45} \psi_y \delta w_y + H_{55} \psi_x \delta w_y) \, dx \, dy
\]

\[
K_{14} = \int (H_{44} \psi_x \phi_x + H_{45} \psi_y \phi_x) \, dx \, dy
\]

\[
K_{15} = \int (H_{45} \psi_x \phi_x + H_{55} \psi_y \phi_x) \, dx \, dy
\]

\[
K_{22} = \int (H_{44} \psi_x \delta w_x + D_{11} \psi_x \delta \phi_x + D_{12} \psi_y \delta \phi_x + D_{13} \psi_x \delta \phi_y + D_{14} \psi_y \delta \phi_y) \, dx \, dy
\]

\[
K_{23} = \int (H_{45} \psi_y \delta w_y + D_{12} \psi_x \delta \phi_x + D_{13} \psi_y \delta \phi_x + D_{14} \psi_x \delta \phi_y + D_{15} \psi_y \delta \phi_y) \, dx \, dy
\]

\[
K_{24} = \int (B_{11} \psi_x \phi_x + B_{13} \psi_y \phi_x + B_{14} \psi_x \phi_y + B_{15} \psi_y \phi_y) \, dx \, dy
\]

\[
K_{25} = \int (B_{12} \psi_x \phi_x + B_{13} \psi_y \phi_x + B_{14} \psi_x \phi_y + B_{15} \psi_y \phi_y) \, dx \, dy
\]

\[
K_{33} = \int (H_{44} \psi_x \delta w_x + H_{45} \psi_y \delta w_x + H_{54} \psi_x \delta w_y + H_{55} \psi_y \delta w_y) \, dx \, dy
\]

\[
K_{34} = \int (H_{44} \psi_x \phi_x + H_{45} \psi_y \phi_x + H_{54} \psi_x \phi_y + H_{55} \psi_y \phi_y) \, dx \, dy
\]

\[
K_{35} = \int (H_{45} \psi_x \phi_x + H_{55} \psi_y \phi_x + H_{54} \psi_x \phi_y + H_{55} \psi_y \phi_y) \, dx \, dy
\]

\[
K_{44} = \int (H_{44} \psi_x \delta w_x + H_{45} \psi_y \delta w_x + H_{54} \psi_x \delta w_y + H_{55} \psi_y \delta w_y) \, dx \, dy
\]

\[
K_{45} = \int (H_{45} \psi_y \delta w_y + H_{55} \psi_x \delta w_y) \, dx \, dy
\]

\[
K_{55} = \int (H_{45} \psi_y \delta w_y + H_{55} \psi_x \delta w_y) \, dx \, dy
\]
\[ K_2^N = \int \left( B_{12} v_x \frac{\partial \phi_x}{\partial x} + B_{13} v_y \frac{\partial \phi_x}{\partial y} + B_{23} v_y \frac{\partial \phi_y}{\partial y} + B_{33} v_y \frac{\partial \phi_y}{\partial x} \right) \, dx \, dy \]

\[ K_3^N = \int \left( B_{12} v_x \frac{\partial \phi_y}{\partial y} + B_{13} v_y \frac{\partial \phi_y}{\partial y} + B_{23} v_y \frac{\partial \phi_x}{\partial x} + B_{33} v_y \frac{\partial \phi_x}{\partial x} \right) \, dx \, dy \]

\[ K_4^N = \int \left( \frac{1}{2} \left( B_{12} v_x \frac{\partial \phi_y}{\partial y} + B_{13} v_y \frac{\partial \phi_y}{\partial y} + B_{23} v_y \frac{\partial \phi_x}{\partial x} + B_{33} v_y \frac{\partial \phi_x}{\partial x} \right) \right) \, dx \, dy \]

\[ K_5^N = \int \left( B_{12} v_x \frac{\partial \phi_y}{\partial y} + B_{13} v_y \frac{\partial \phi_y}{\partial y} + B_{23} v_y \frac{\partial \phi_x}{\partial x} + B_{33} v_y \frac{\partial \phi_x}{\partial x} \right) \, dx \, dy \]

\[ K_6^N = \int \left( B_{12} v_x \frac{\partial \phi_y}{\partial y} + B_{13} v_y \frac{\partial \phi_y}{\partial y} + B_{23} v_y \frac{\partial \phi_x}{\partial x} + B_{33} v_y \frac{\partial \phi_x}{\partial x} \right) \, dx \, dy \]

nonlinear coefficients
\[ K_{11}^{NL} = \int_0^1 \left( \frac{1}{2} w_{,x} (B_{12} w_{,x} \phi_{,x} + B_{32} w_{,x} \phi_{,x}) + B_{13} w_{,x} \phi_{,x} + B_{33} w_{,x} \phi_{,x} \right) \, dx \, dy \]
\[ + \frac{1}{2} w_{,x} (B_{12} w_{,x} \phi_{,x} + B_{32} w_{,x} \phi_{,x} + B_{11} w_{,x} \phi_{,x} + B_{31} w_{,x} \phi_{,x}) \, dx \, dy \]

\[ K_{22}^{NL} = \int_0^1 \left( \frac{1}{2} w_{,x} (B_{12} w_{,x} \phi_{,x} + B_{32} w_{,x} \phi_{,x}) + B_{13} w_{,x} \phi_{,x} + B_{33} w_{,x} \phi_{,x} \right) \, dx \, dy \]
\[ + \frac{1}{2} w_{,x} (B_{12} w_{,x} \phi_{,x} + B_{32} w_{,x} \phi_{,x} + B_{13} w_{,x} \phi_{,x} + B_{33} w_{,x} \phi_{,x}) \, dx \, dy \]

\[ K_{31}^{NL} = \int_0^1 \left( \frac{1}{2} w_{,x} (A_{11} w_{,x} \phi_{,x} + A_{13} w_{,x} \phi_{,x} + A_{31} w_{,x} \phi_{,x} + A_{33} w_{,x} \phi_{,x}) \right) \, dx \, dy \]
\[ + \frac{1}{2} w_{,x} (B_{12} w_{,x} \phi_{,x} + B_{32} w_{,x} \phi_{,x} + B_{13} w_{,x} \phi_{,x} + B_{33} w_{,x} \phi_{,x}) \, dx \, dy \]

\[ K_{32}^{NL} = \int_0^1 \left( \frac{1}{2} w_{,x} (A_{21} w_{,x} \phi_{,x} + A_{23} w_{,x} \phi_{,x} + A_{31} w_{,x} \phi_{,x} + A_{33} w_{,x} \phi_{,x}) \right) \, dx \, dy \]
\[ + \frac{1}{2} w_{,x} (A_{22} w_{,x} \phi_{,x} + A_{24} w_{,x} \phi_{,x} + A_{32} w_{,x} \phi_{,x} + A_{34} w_{,x} \phi_{,x}) \, dx \, dy \]

REFERENCES