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Periodic solutions to a non-linear differential equation of the order $2n + 1$


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**Equazioni differenziali ordinarie.** — Periodic solutions to a non-linear differential equation of the order $2n + 1$. Nota (*) di MONIKA KUBICOVA, presentata dal Corrisp. R. Conti.

**Abstract.** — A criterion for the existence of periodic solutions of an ordinary differential equation of order $k$ proved by J. Andres and J. Voráček for $k = 3$ is extended to an arbitrary odd $k$.

**Key words:** Nonlinear ordinary differential equations; Periodic solutions; Existence.

**Riassunto.** — Si estende ad una equazione differenziale ordinaria di ordine dispari arbitrario $k$ un criterio di esistenza di soluzioni periodiche dimostrato da J. Andres e J. Voráček per il caso di $k = 3$.

We consider the equation:

\[(1) \quad x^{(2n+1)} + a_1(t,x,x',\ldots,x^{(2n)}) x^{(n+1)} + \ldots + a_{n+1}(t,x,x',\ldots,x^{(2n)}) x' + b(x) = e(t,x,x',\ldots,x^{(2n)})\]

with functions $a_1, \ldots, a_{n+1}, e \in C(R^{2n+2}, R)$, $b \in C(R, R)$. Furthermore we assume the $\omega$-periodicity in the variable $t$ of $a_1, \ldots, a_{n+1}, e$ ($\omega > 0$, $n \geq 1$).

The existence of an $\omega$-periodic solution of (1) will be proved by the Leray-Schauder fixed point technique. In the paper the result of J. Andres and J. Voráček from [1] for the third order equations to the equations of the order $2n + 1$ is extended.

The equation (1) results for $p = 1$ from the following family of differential equations depending on the parameter $p$:

\[(2_p) \quad x^{(2n+1)} + p[a_1(t,x,x',\ldots,x^{(2n)}) x^{(n+1)} + \ldots + a_{n+1}(t,x,x',\ldots,x^{(2n)}) x'] + (b(x) - cx)] + cx = pe(t,x,x',\ldots,x^{(2n)})\]

with a suitable constant $c \neq 0$.

For the existence of an $\omega$-periodic solution of (1) the following conditions are sufficient:

(i) All the $\omega$-periodic solutions $x(t)$ of $(2_p)$ and $x'(t), \ldots, x^{(2n)}(t)$ are for $0 \leq p \leq 1$ a priori bounded by a constant independent of $p$.

(ii) The linear equation:

\[(2_0) \quad x^{(2n+1)} + cx = 0\]

(resulting from $(2_p)$ for $p = 0$) has no $\omega$-periodic solution different from identical zero.

The condition (ii) is satisfied if and only if the characteristic equation of $(2_0)$, i.e. the equation

\[z^{2n+1} + c = 0\]

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has no root in the form $i(2\pi/c)k$, $k$ is an integer. It is clear that the condition (ii) is satisfied for any $c \neq 0$. Hence it is sufficient to consider the case when the condition (i) is satisfied.

From the Wirtinger Lemma [2], p. 185, it follows that for every $\omega$-periodic function $x(t) \in C^2(R, R)$ such that $x^{(2n+1)} \in L^2(t, t + \omega)$ we have:

$$
(3) \quad \int_t^{t+\omega} (x^{(k)}(s))^2 \, ds \leq \left( \frac{\omega}{2\pi} \right)^2 \int_t^{t+\omega} (x^{(0)}(s))^2 \, ds, \quad \text{for all } t \in R \text{ and } k = 2, 3, \ldots, 2n + 1.
$$

In the following the composited function $a_k(t, x(t), x'(t), \ldots, x^{(2n)}(t))$ of the variable $t$ formed by the function $a_k$ and the function $x(t)$ will be denoted by the symbol $a_k(t)$ ($k = 1, 2, \ldots, n + 1$).

In the same sense we use the symbols $b_k(t)$, $e_k(t)$. Further we put:

$$
\omega_1 = \frac{\omega}{2\pi}.
$$

At first we prove the estimates for $\omega$-periodic solutions of $(2_p)$ in the $L^2$ space norm.

**Lemma 1.** If the following inequalities:

$$
|a_k(t, x_1, \ldots, x_{2n})| \leq A_k, \quad \text{for } k = 1, 2, \ldots, n + 1,
$$

$$
|e(t, x_1, \ldots, x_{2n})| \leq E
$$

hold for all $t, x_1, x_2, \ldots, x_{2n}$, and

$$
\theta := 1 - \sum_{k=1}^{n+1} A_k \omega_1^{n+k-1} > 0,
$$

then every $\omega$-periodic solution $x(t)$ of $(2_p)$ satisfies the inequality:

$$
(4) \quad \int_t^{t+\omega} (x^{(n+1)}(s))^2 \, ds \leq D_{n+1}^2, \quad \text{for all } t,
$$

where

$$
D_{n+1}^2 := \left[ \frac{\omega_1^2 E^2}{\theta} \right] \omega.
$$

**Proof.** Substituting a fixed $x(t)$ into $(2_p)$, multiplying the obtained identity by $x'(t)$ and integrating in $\langle t, t + \omega \rangle$ we come to:

$$
(-1)^n \int_t^{t+\omega} (x^{(n+1)}(s))^2 \, ds = p \int_t^{t+\omega} \left( -a_{nk}(s) x^{(n+1)}(s) x'(s) - \cdots - a_{n+1k}(s) x'(s)^2 + e_{nk}(s) x'(s) \right) \, ds.
$$

By the assumptions of Lemma 1 we get:

$$
\int_t^{t+\omega} (x^{(n+1)}(s))^2 \, ds \leq \int_t^{t+\omega} \left[ A_1 |x^{(n+1)}(s)| x'(s)| + \cdots + A_{n+1} x'(s)^2 + E |x'(s)| \right] \, ds.
$$

Using (3) and the Schwarz inequality we get the inequality:

$$
\theta^2 \int_t^{t+\omega} (x^{(n+1)}(s))^2 \, ds \leq E^2 \omega_1^n \omega,
$$

for all $t$ from where we come to (4).
**Corollary 1.** If all assumptions of Lemma 1 are fulfilled, then for every \( \omega \)-periodic solution \( x(t) \) of (2) we have:

\[
\int_{t}^{t+\omega} (x^{(k)}(s))^2 \, ds \leq D_k^2
\]

for \( k = n + 1, n, \ldots, 2, 1 \) and for all \( t \in R \), where

\[
D_k := \omega_1 D_{k+1}
\]

for \( k = n, n - 1, \ldots, 2, 1 \),

\[
|x^{(k)}(s)| \leq D_k^2
\]

for \( k = 1, 2, \ldots, n \) and for all \( s \in R \), where \( D_k^* := \sqrt{\omega_1 D_{k+1}} \).

**Proof.** By the finite induction we get the estimates (5) from (3) and (4). Furthermore for \( k = 0, 1, \ldots, n - 1 \) and for all \( t \) the \( \omega \)-periodic function \( x^{(k)}(s) \) fulfils on \( (t, t + \omega) \) the assumptions of the Mean value theorem. Thus there is a point \( t_i \in (t, t + \omega) \) such that:

\[
x^{(k+1)}(t_i) = 0.
\]

Consequently

\[
x^{(k+1)}(s) = \int_{t_i}^{s} x^{(k+2)}(u) \, du
\]

for \( k = 0, 1, \ldots, n - 1 \) and for all \( s \in (t, t + \omega) \).

On the basis of the Schwarz inequality and (5) we get the estimate (6) in the sup norm.

**Lemma 2.** If all assumptions of Lemma 1 are fulfilled and there exist real numbers \( c \neq 0, m > 0 \) such that the inequality

\[
x^b(x) \operatorname{sgn} c \geq |c|x^2
\]

is true, then every \( \omega \)-periodic solution \( x(t) \) of (2) satisfies

\[
|x(t)| \leq D_0',
\]

for all \( t \in R \) with \( D_0' := R + D_1' \omega \)

\[
R := \max \left\{ m, \frac{A_1 D_{n+1} + \ldots + A_n D_1 + E}{c} \right\}.
\]

**Proof.** We again substitute \( x(t) \) into (2). Multiplying the resulting identity by \( x(t) \) and integrating in \( (t, t + \omega) \) we obtain for every \( t \):

\[
p \int_{t}^{t+\omega} \left( a_{1x}(s) x^{(n+1)}(s) x(s) + \ldots + a_{n+1x}(s) x'(s) x(s) - e_{x}(s) x(s) \right) \, ds =
\]

\[
= (p - 1) \int_{t}^{t+\omega} c x^2(s) \, ds - p \int_{t}^{t+\omega} x(s) b_{x}(s) \, ds.
\]

Hence using the assumptions of Lemma 1 we get the inequality:

\[
\int_{t}^{t+\omega} ((1 - p) c x^2(s) + px(s) b_{x}(s) \operatorname{sgn} c) \, ds \leq
\]

\[
\leq \int_{t}^{t+\omega} \left[ A_1|x^{(n+1)}(s)| x(s)| + \ldots + A_{n+1}|x'(s) x(s)| + E|x(s)| \right] \, ds.
\]

If on the whole interval \( (t, t + \omega) \) the inequality \( |x(s)| > R \geq m \) held, by (7) and (9) we
would have:

\[ c \int_t^{t+\omega} x^2(s) \, ds \leq \int_t^{t+\omega} \left[ A_1|x^{n+1}(s)| + \ldots + A_{n+1}|x'(s)| + E|x(s)| \right] \, ds \]

and on the basis of the Schwarz inequality and (7) this would imply the inequality:

\[ c^2 \int_t^{t+\omega} x^2(s) \, ds \leq c^2 R^2 \omega . \]

On the other hand by the inequality \(|x(s)| > R\) on \(\langle t, t + \omega \rangle\) we come to

\[ c^2 \int_t^{t+\omega} x^2(s) \, ds > c^2 R^2 \omega \]

which contradicts (10).

Thus on each interval \(\langle t, t + \omega \rangle\) there must exist a point \(t_1\) with

\[ |x(t_1)| \leq R . \]

Using the Mean value formula we get for all \(s \in \langle t, t + \omega \rangle:\)

\[ |x(s)| = |x(t_1)| + |x'(t_2)| |s - t_1| . \]

The periodicity of \(x(t)\) assures that (8) holds for all \(t \in R\).

**Lemma 3.** If all assumptions of Lemma 2 are satisfied then denoting

\[ H := \max_{x \in D_0}|h(x)| \]

we have for every \(\omega\)-periodic solution \(x(t)\) of \((2p)\)

\[ \int_t^{t+\omega} (x^{2n+1}(s))^2 \, ds \leq D_{2n+1}^2 , \]

for all \(t \in R\),

where \(D_{2n+1} := A_1D_{n+1} + \ldots + A_{n+1}D_1 + (E + H) \sqrt{\omega} \).

**Proof.** We again substitute \(x(t)\) into \((2p)\). Multiplying the obtained identity by \(x^{2n+1}(t)\) and integrating in \(\langle t, t + \omega \rangle\) we get the identity:

\[ \int_t^{t+\omega} (x^{2n+1}(s))^2 \, ds = -\int_t^{t+\omega} \left[ a_{1n}(s) x^{n+1}(s) + \ldots + a_{n+1n}(s) x'(s) + b_n(s) - e_n(s) \right] x^{2n+1}(s) \, ds . \]

Using the assumptions of Lemma 3 we come to:

\[ \int_t^{t+\omega} (x^{2n+1}(s))^2 \, ds \leq \int_t^{t+\omega} \left[ A_1|x^{n+1}(s)| x^{2n+1}(s)| + \ldots + A_{n+1}|x'(s)| x^{2n+1}(s)| + E|x^{2n+1}(s)| + H|x^{2n+1}(s)| \right] \, ds . \]

Hence the Schwarz inequality and (5) implies that:

\[ \int_t^{t+\omega} (x^{2n+1}(s))^2 \, ds \leq (A_1D_{n+1} + \ldots + A_{n+1}D_1 + (E + H) \sqrt{\omega})^2 := D_{2n+1}^2 . \]

Using (11) and (3) we can extend the inequalities (5) for \(k = 2n + 1, \ldots, 2, 1\). Then by a similar method as in Corollary 1, the estimates (6) can be extended for \(k = 1, 2, \ldots, 2n\). Thus the following corollary holds.
**Corollary 2.** If all assumptions of Lemma 2 are fulfilled, then every $\omega$-periodic solution $x(t)$ of (2$\omega$) satisfies:

\[(12) \quad \int_{t}^{t+\omega} (x^{(k)}(s))^2 \, ds \leq D_k^2 \quad \text{for } k = 2n + 1, 2n, \ldots, 2, 1, \text{ and for all } t \in \mathbb{R},\]

where $D_{2n+1}$ is given in Lemma 3

\[D_k = \omega_1 D_{k+1} \quad \text{for } k = 2n, 2n - 1, \ldots, n + 2,\]

$D_{n+1}$ is given in Lemma 1

\[D_k = \omega_1 D_{k+1} \quad \text{for } k = n, n - 1, \ldots, 1.\]

Further:

\[(13) \quad |x^{(k)}(s)| \leq D_k' \quad \text{for } k = 1, 2, \ldots, 2n, \text{ and for all } t \in \mathbb{R},\]

where

\[D_k' = \sqrt{\omega} D_{k+1} \quad \text{for } k = 1, 2, \ldots, 2n.\]

**Theorem.** If the following inequalities:

\[|a_k(t, x_1, \ldots, x_{2n})| \leq A_k \quad \text{for } k = 1, 2, \ldots, n + 1,\]

\[|e(t, x_1, \ldots, x_{2n})| \leq E\]

hold for all $t, x_1, x_2, \ldots, x_{2n},$

\[\theta := 1 - \sum_{k=1}^{n+1} A_k \omega_1^{s+k-1} > 0,\]

and if there exist such real numbers $c \neq 0, m > 0$ that (7) holds, then the equation (1) admits an $\omega$-periodic solution.

**Proof.** From Corollary 2 we get for every $\omega$-periodic solution $x(t)$ of (2$\omega$):

\[\sum_{k=0}^{2n} |x^{(k)}(t)| \leq \sum_{k=0}^{2n} D_k = P,\]

with $P$ independent of $p \in (0, 1)$.

Thus both conditions (i), (ii) which are sufficient for the existence of an $\omega$-periodic solution of (1), are fulfilled.

**References**
