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Equazioni differenziali ordinarie. — *Periodic solutions to a non-linear differential equation of the order* $2n+1$. Nota (*) di MONIKA KUBICOVA, presentata dal Corrisp. R. CONTI.

ABSTRACT. — A criterion for the existance of periodic solutions of an ordinary differential equation of order k proved by J. Andres and J. Voráček for $k=3$ is extended to an arbitrary odd k.

KEY WORDS: Nonlinear ordinary differential equations; Periodic solutions; Existence.

RIASSUNTO. — Si estende ad una equazione differenziale ordinaria di ordine dispari arbitrario k un criterio di esistenza di soluzioni periodiche dimostrato da J. Andres e J. Voráček per il caso di $k=3$.

We consider the equation:

$$(1)\qquad x^{(2n+1)} + a_1(t,x,x',\ldots,x^{(2n)})\,x^{(n+1)} + \ldots + a_{n+1}(t,x,x',\ldots,x^{(2n)})\,x' + h(x) = = e(t,x,x',\ldots,x^{(2n)})$$

with functions $a_1,\ldots,a_{n+1}$, $e \in C(R^{2n+2},R)$, $h \in C(R,R)$. Furthermore we assume the ω-periodicity in the variable t of $a_1,\ldots,a_{n+1}$, e $(\omega>0,\ n\geqslant 1)$.

The existence of an ω-periodic solution of (1) will be proved by the Leray-Schauder fixed point technique. In the paper the result of J. Andres and J. Voráček from [1] for the third order equations to the equations of the order $2n+1$ is extended.

The equation (1) results for $p:=1$ from the following family of differential equations depending on the parameter p:

$$(2_p)\qquad x^{(2n+1)} + p[a_1(t,x,x',\ldots,x^{(2n)})\,x^{(n+1)} + \ldots + a_{n+1}(t,x,x',\ldots,x^{(2n)})\,x' + + (h(x)-cx)] + cx = pe(t,x,x',\ldots,x^{(2n)})$$

with a suitable constant $c \neq 0$.

For the existence of an ω-periodic solution of (1) the following conditions are sufficient:

(i) All the ω-periodic solutions $x(t)$ of (2_p) and $x'(t),\ldots,x^{(2n)}(t)$ are for $0 \leqslant p \leqslant 1$ a priori bounded by a constant independent of p.

(ii) The linear equation:

$$(2_0)\qquad x^{(2n+1)} + cx = 0$$

(resulting from (2_p) for $p=0$) has no ω-periodic solution different from identical zero.

The condition (ii) is satisfied if and only if the characteristic equation of (2_0), *i.e.* the equation

$$z^{2n+1} + c = 0$$

(*) Pervenuta all'Accademia il 17 agosto 1988.

has no root in the form $i(2\pi/\omega)k$, k is an integer. It is clear that the condition (ii) is satisfied for any $c \neq 0$. Hence it is sufficient to consider the case when the condition (i) is satisfied.

From the Wirtinger Lemma [2], p. 185, it follows that for every ω-periodic function $x(t) \in C^{2n}(R,R)$ such that $x^{(2n+1)} \in L^2(\langle t, t+\omega\rangle)$ we have:

$$(3) \quad \int_t^{t+\omega} (x^{(k-1)}(s))^2\, ds \leq \left(\frac{\omega}{2\pi}\right)^2 \int_t^{t+\omega} (x^{(k)}(s))^2\, ds, \qquad \text{for all } t \in R \text{ and } k = 2, 3, \ldots, 2n+1.$$

In the following the composited function $a_k(t, x(t), x'(t), \ldots, x^{(2n)}(t))$ of the variable t formed by the function a_k and the function $x(t)$ will be denoted by the symbol $a_{kx}(t)$ $(k = 1, 2, \ldots, n+1)$.

In the same sense we use the symbols $h_x(t)$, $e_x(t)$. Further we put:

$$\omega_1 := \frac{\omega}{2\pi}.$$

At first we prove the estimates for ω-periodic solutions of (2_p) in the L^2 space norm.

Lemma 1. If the following inequalities:

$$|a_k(t, x_1, \ldots, x_{2n})| \leq A_k, \qquad \text{for } k = 1, 2, \ldots, n+1,$$
$$|e(t, x_1, \ldots, x_{2n})| \leq E$$

hold for all t, $x_1, x_2, \ldots, x_{2n}$, and

$$\theta := 1 - \sum_{k=1}^{n+1} A_k \omega_1^{n+k-1} > 0,$$

then every ω-periodic solution $x(t)$ of (2_p) satisfies the inequality:

$$(4) \quad \int_t^{t+\omega} (x^{(n+1)}(s))^2\, ds \leq D_{n+1}^2, \qquad \text{for all } t,$$

where

$$D_{n+1}^2 := \left[\frac{\omega_1^n E}{\theta}\right]^2 \omega.$$

Proof. Substituting a fixed $x(t)$ into (2_p), multiplying the obtained identity by $x'(t)$ and integrating in $\langle t, t+\omega\rangle$ we come to:

$$(-1)^n \int_t^{t+\omega} (x^{(n+1)}(s))^2\, ds = p \int_t^{t+\omega} (-a_{1x}(s)\, x^{(n+1)}(s)\, x'(s) - \ldots - a_{n+1x}(x'(s))^2 + e_x(s)\, x'(s))\, ds.$$

By the assumptions of Lemma 1 we get:

$$\int_t^{t+\omega} (x^{(n+1)}(s))^2\, ds \leq \int_t^{t+\omega} [A_l |x^{(n+1)}(s)\, x'(s)| + \ldots + A_{n+1}(x'(s))^2 + E|x'(s)|]\, ds.$$

Using (3) and the Schwarz inequality we get the inequality:

$$\theta^2 \int_t^{t+\omega} (x^{(n+1)}(s))^2\, ds \leq E^2 \omega_1^{2n} \omega, \qquad \text{for all } t$$

from where we come to (4).

Corollary 1. If all assumptions of Lemma 1 are fulfilled, then for every ω-periodic solution $x(t)$ of (2_p) we have:

$$(5)\qquad \int_t^{t+\omega} (x^{(k)}(s))^2\, ds \leqslant D_k^2 \qquad \text{for } k=n+1, n, \ldots, 2, 1 \text{ and for all } t \in R,$$

where

$$D_k := \omega_1 D_{k+1} \qquad \text{for } k=n, n-1, \ldots, 2, 1,$$

$$(6)\qquad |x^{(k)}(s)| \leqslant D'_k \qquad \text{for } k=1,2,\ldots,n \text{ and for all } s \in R,$$

where $D'_k := \sqrt{\omega}\, D_{k+1}$.

Proof. By the finite induction we get the estimates (5) from (3) and (4). Furthermore for $k=0,1,\ldots,n-1$ and for all t the ω-periodic function $x^{(k)}(s)$ fulfils on $\langle t, t+\omega\rangle$ the assumptions of the Mean value theorem. Thus there is a point $t_1 \in \langle t, t+\omega\rangle$ such that:

$$x^{(k+1)}(t_1) = 0\,.$$

Consequently

$$x^{(k+1)}(s) = \int_{t_1}^{s} x^{(k+2)}(u)\, du \qquad \text{for } k=0,1,\ldots,n-1 \text{ and for all } s \in \langle t, t+\omega\rangle\,.$$

On the basis of the Schwarz inequality and (5) we get the estimate (6) in the sup norm.

Lemma 2. If all assumptions of Lemma 1 are fulfilled and there exist real numbers $c \neq 0$, $m > 0$ such that the inequality

$$(7)\qquad xh(x)\operatorname{sgn} c \geqslant |c| x^2 \qquad \text{for every } |x| \geqslant m$$

is true, then every ω-periodic solution $x(t)$ of (2_p) satisfies

$$(8)\qquad |x(t)| \leqslant D'_0, \qquad \text{for all } t \in R$$

with $D'_0 := R + D'_1 \omega$

$$R := \max\left[m; \frac{A_1 D_{n+1} + \ldots + A_{n+1} D_1}{c} + \frac{E}{c}\right].$$

Proof. We again substitute $x(t)$ into (2_p). Multiplying the resulting identity by $x(t)$ and integrating in $\langle t, t+\omega\rangle$ we obtain for every t:

$$p \int_t^{t+\omega} (a_{1x}(s)\, x^{(n+1)}(s)\, x(s) + \ldots + a_{n+1x}(s)\, x'(s)\, x(s) - e_x(s)\, x(s))\, ds =$$

$$= (p-1) \int_t^{t+\omega} c x^2(s)\, ds - p \int_t^{t+\omega} x(s)\, h_x(s)\, ds\,.$$

Hence using the assumptions of Lemma 1 we get the inequality:

$$(9)\qquad \int_t^{t+\omega} ((1-p)\, c x^2(s) + p x(s)\, h_x(s)\operatorname{sgn} c)\, ds \leqslant$$

$$\leqslant \int_t^{t+\omega} [A_1 |x^{(n+1)}(s)\, x(s)| + \ldots + A_{n+1} |x'(s)\, x(s)| + E|x(s)|]\, ds\,.$$

If on the whole interval $\langle t, t+\omega\rangle$ the inequality $|x(s)| > R$ $(\geqslant m)$ held, by (7) and (9) we would have:

$$c\int_t^{t+\omega} x^2(s)\,ds \leq \int_t^{t+\omega} [A_1|x^{(n+1)}(s)\,x(s)| + \ldots + A_{n+1}|x'(s)\,x(s)| + E|x(s)|]\,ds$$

and on the basis of the Schwarz inequality and (7) this would imply the inequality:

$$c^2\int_t^{t+\omega} x^2(s)\,ds \leq c^2 R^2\,\omega\,. \tag{10}$$

On the other hand by the inequality $|x(s)| > R$ on $\langle t, t+\omega\rangle$ we come to

$$c^2\int_t^{t+\omega} x^2(s)\,ds > c^2 R^2\,\omega$$

which contradicts (10).

Thus on each interval $\langle t, t+\omega\rangle$ there must exist a point t_1 with

$$|x(t_1)| \leq R\,.$$

Using the Mean value formula we get for all $s \in \langle t, t+\omega\rangle$:

$$|x(s)| \leq |x(t_1)| + |x'(t_2)|\,|s - t_1|\,.$$

The periodicity of $x(t)$ assures that (8) holds for all $t \in R$.

LEMMA 3. If all assumptions of Lemma 2 are satisfied then denoting

$$H := \max_{x \leq D_0'} |h(x)|$$

we have for every ω-periodic solution $x(t)$ of (2_p)

$$\int_t^{t+\omega} (x^{(2n+1)}(s))^2\,ds \leq D_{2n+1}^2\,, \qquad \text{for all } t \in R\,, \tag{11}$$

where $D_{2n+1} := A_1 D_{n+1} + \ldots + A_{n+1} D_1 + (E + H)\sqrt{\omega}$.

PROOF. We again substitute $x(t)$ into $(2p)$. Multiplying the obtained identity by $x^{(2n+1)}(t)$ and integrating in $\langle t, t+\omega\rangle$ we get the identity:

$$\int_t^{t+\omega} (x^{(2n+1)}(s))^2\,ds = -p\int_t^{t+\omega} (a_{1x}(s)\,x^{(n+1)}(s) + \ldots + a_{n+1x}(s)\,x'(s) + h_x(s) - e_x(s))\,x^{(2n+1)}(s)\,ds\,.$$

Using the assumptions of Lemma 3 we come to:

$$\int_t^{t+\omega} (x^{(2n+1)}(s))^2\,ds \leq \int_t^{t+\omega} [A_1|x^{(n+1)}(s)\,x^{(2n+1)}(s)| + \ldots +$$
$$+ A_{n+1}|x'(s)\,x^{(2n+1)}(s)| + E|x^{(2n+1)}(s)| + H|x^{(2n+1)}(s)|]\,ds\,.$$

Hence the Schwarz inequality and (5) implies that:

$$\int_t^{t+\omega} (x^{(2n+1)}(s))^2\,ds \leq (A_1 D_{n+1} + \ldots + A_{n+1} D_1 + (E + H)\sqrt{\omega})^2 := D_{2n+1}^2\,.$$

Using (11) and (3) we can extend the inequalities (5) for $k = 2n+1, \ldots, 2, 1$. Then by a similar method as in Corollary 1, the estimates (6) can be extended for $k = 1, 2, \ldots, 2n$. Thus the following corollary holds.

COROLLARY 2. If all assumptions of Lemma 2 are fulfilled, then every ω-periodic solution $x(t)$ of (2_p) satisfies:

$$(12) \qquad \int_t^{t+\omega} (x^{(k)}(s))^2\, ds \leq D_k^2 \qquad \text{for } k = 2n+1, 2n, \ldots, 2, 1, \text{ and for all } t \in R,$$

where D_{2n+1} is given in Lemma 3

$D_k := \omega_1 D_{k+1}$ for $k = 2n, 2n-1, \ldots, n+2$,

D_{n+1} is given in Lemma 1

$D_k := \omega_1 D_{k+1}$ for $k = n, n-1, \ldots, 1$.

Further:

$$(13) \qquad |x^{(k)}(s)| \leq D'_k \qquad \text{for } k = 1, 2, \ldots, 2n, \text{ and for all } t \in R,$$

where

$$D'_k = \sqrt{\omega}\, D_{k+1} \qquad \text{for } k = 1, 2, \ldots, 2n.$$

THEOREM. If the following inequalities:

$$|a_k(t, x_1, \ldots, x_{2n})| \leq A_k \qquad \text{for } k = 1, 2, \ldots, n+1,$$

$$|e(t, x, \ldots, x_{2n})| \leq E$$

hold for all $t, x_1, x_2, \ldots, x_{2n}$,

$$\theta := 1 - \sum_{k=1}^{n+1} A_k \omega_1^{n+k-1} > 0,$$

and if there exist such real numbers $c \neq 0$, $m > 0$ that (7) holds, then the equation (1) admits an ω-periodic solution.

PROOF. From Corollary 2 we get for every ω-periodic solution $x(t)$ of (2_p):

$$\sum_{k=0}^{2n} |x^{(k)}(t)| \leq \sum_{k=0}^{2n} D'_k := P,$$

with P independent of $p \in \langle 0, 1 \rangle$.

Thus both conditions (i), (ii) which are sufficient for the existence of an ω-periodic solution of (1), are fulfilled.

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