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Perturbation results for a class of singular Hamiltonian systems

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Equazioni differenziali. — *Perturbation results for a class of singular Hamiltonian systems.* Nota di ANTONIO AMBROSETTI (*) e IVAR EKELAND (**), presentata (***) dal Corrisp. A. AMBROSETTI.

ABSTRACT. — The existence of solutions with prescribed period T for a class of Hamiltonian systems with a Keplerian singularity is discussed.

KEY WORDS: Hamiltonian systems; Singular nonlinearities; Critical point theory.

RIASSUNTO. — *Risultati di perturbazione per una classe di sistemi Hamiltoniani con singolarità.* Viene discussa l'esistenza di soluzioni di periodo assegnato T per una classe di sistemi Hamiltoniani con singolarità di tipo Kepleriano.

1. INTRODUCTION.

This note deals with the existence of T-periodic solutions of a class of Hamiltonian systems with n degrees of freedom

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \tag{1}$$

where the Hamiltonian $H = H(t,p,q)$ is T-periodic in t and has a singularity at $q = 0$.

In some recent papers the case in which $H = (1/2)|p|^2 + V(t,q)$ and $V(t,\cdot)$ behaves like $-1/|q|^\alpha$ near $q = 0$ has been investigated. More precisely, in [1] the existence of infinitely many T-periodic solutions not crossing the singularity $q = 0$ (non-collision orbits) is proved, under the main restriction that $\alpha \geqslant 2$. On the other hand, in [2] V has been taken of the more particular form: $V = -1/|q|^\alpha + \varepsilon U(t,q)$. In the last case any $\alpha > 0$ is allowed (when $\alpha = 1$ U is supposed to even in q and $(T/2)$-periodic in t) and it is shown that T-periodic solutions «branch off» from the circular orbits of the unperturbed system $q'' + \alpha(q/|q|^{\alpha+2}) = 0$.

In all the preceding papers (1) reduces to the second order system

$$\frac{d^2 q}{dt^2} + \frac{\partial V}{\partial q} = 0, \tag{2}$$

whose non-collision orbits are the critical points of the functional

$$f: H^1(S^1, \boldsymbol{R}^n - \{0\}) \to \boldsymbol{R}, \quad f(q) = \int_0^T \left\{\frac{1}{2}|q|^2 - V(t,q)\right\} dt,$$

which is suitable either for the Morse theory or for some perturbation results in critical point theory [3]. For other results concerning second order systems with singular potentials we refer, for example to [5, 6, 7].

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In the present note we outline the arguments which permit the extension of the results of [2] to a class of *first order* Hamiltonian System such as (1). A more complete discussion as well as detailed proofs are contained in a forthcoming paper [4].

2. The main existence results.

Set $\Omega = \boldsymbol{R}^n \times \boldsymbol{R}^n - (\boldsymbol{R}^n \times \{0\})$ and let K_ε, $H_\varepsilon \in C^2(\boldsymbol{R} \times \Omega, \boldsymbol{R})$ be given by

$$K_\varepsilon(t,p,q) = \frac{1}{2}|p|^2 - \phi(|q|) + \varepsilon W(t,p,q), \tag{3}$$

$$H_\varepsilon(t,p,q) = \frac{1}{2}(Az,z) + K_\varepsilon(t,p,q), \tag{4}$$

where $z = (p,q) \in \Omega$, $(\cdot,\cdot)$ denotes the Euclidean scalar product, A is a symmetric matrix in $\boldsymbol{R}^{2n}$, W is (smooth and) T-periodic in t and $\phi \in C^1(\boldsymbol{R}^+;\boldsymbol{R}^+)$ is convex and $\phi(r) \to +\infty$ as $r \to 0+$.

We denote by J the symplectic matrix $J(p,q) = (-q,p)$ and look for T-periodic solutions of the Hamiltonian System

$$-J\frac{dz}{dt} = \nabla H_\varepsilon(t,z). \tag{5}$$

near those of the unperturbed autonomous system

$$-J\frac{dz}{dt} = \nabla H_0(z). \tag{6}$$

Let $E = C(\boldsymbol{R}/T\mathbb{Z};\boldsymbol{R}^{2n})$, Z denote a set of T-periodic solutions of (6) such that $\Omega \supset B = B_Z =: \{z(t) | t \in Z, z \in Z\}$. For $z \in Z$ we denote by $Y(z)$ the set of the T-periodic solutions of

$$-J\frac{dy}{dt} = H_0''(z(t))\, y,$$

and suppose that:

(Z) Z is a C^1-submanifold of E, and for the tangent space $T_z Z$ there results: $Y(Z) = T_z Z \ \forall z \in Z$.

Let cat$(\cdot)$ denote the Lusternik-Schnirelman category. Our main existence result is:

Theorem 1. *Let us suppose that* (4) *and* (Z) *hold. Moreover we assume that*

(*A*) *the matrix A has no eigenvalue of the form $ik\omega$, $\omega = 2\pi/T$, $k \in \mathbb{Z}$.*

Then there exist a neighbourhood N' of Z and $\varepsilon' > 0$ such that $\forall |\varepsilon| < \varepsilon'$ (5) *has at least* cat(Z) *T-periodic solutions in N'.*

The proof of Theorem 1 is based on a *local variational principle* suitable for the application of the perturbation results of [3]. For this let us start with some preliminary lemmas.

Set $B^* = \nabla K_0(B). = \{z^* = (p^*,q^*) \in \Omega | p^* = p, q^* = -\phi'(|q|)(q/|q|), (p,q) \in B\}$ and denote by ϕ^* the Legendre transform of ϕ. It is possible to prove:

Lemma 2. *There exist $\varepsilon^* > 0$, a neighbourhood U of B^* in Ω and $W^* \in C^2(\{|\varepsilon| < \varepsilon^*\} \times \boldsymbol{R} \times U;\boldsymbol{R})$ such that, letting $K(\varepsilon,t,z^*) = (1/2)|p^*|^2 - \phi^*(|q^*|) - W^*(\varepsilon,t,z^*)$, there re-*

sults:

$$K^*(0,z^*) \equiv \frac{1}{2}|p^*|^2 - \phi^*(|q^*|);$$

$$\nabla K^*(\varepsilon,t,z^*) = z \quad \textit{iff} \quad z \in \Omega \quad \textit{and} \quad \nabla K_\varepsilon(t,z) = z^*.$$

Next, we note that assumption (A) allows us to define a linear, compact, selfadjoint operator $L: E \to E$ by setting

$$z = Lv \quad \text{iff} \quad -J\frac{dz}{dt} - Az = v.$$

Let $N = \{u \in E | u(t) \in U\}$ and define $F \in C^2(\{|\varepsilon| < \varepsilon^*\} \times N; \boldsymbol{R})$ by setting

$$F(\varepsilon,u) = \int_0^T \left\{\frac{1}{2}(Lu,u) - K^*(\varepsilon,t,u)\right\} dt.$$

We are now in position to state our Local Variational Principle:

LEMMA 3. *If* $u \in N$ *is a critical point of* $F(\varepsilon,\cdot)$ *then* $z = \nabla K^*(\varepsilon,t,u) \in B$ *is a T-periodic solution of* (5).

Note that the Local Variational Principle stated above fits into the framework of the so called Dual Action Principle. In fact, the Clarke's Dual Principle cannot be applied directly here because H_ε is singular and convex in p but concave in q.

Lemma 3 leads to seek for critical points of F in N. This will be done by using the perturbation theory developed in [3]. Set $Z^* = \nabla K_0(Z)$ and $F = F(0,\cdot)$.

LEMMA 4. *If (Z) holds then* Z^* *is non-degenerate critical manifold for F, in the sense that* $\nabla F(u) = 0$ *and* $T_u Z^* = \operatorname{Ker}\{F''(u)\}$ *for all* $u \in Z^*$.

Lemma 4 enables us to apply the results of [3] which yield the existence of cat (Z^*) $(=\text{cat}(Z))$ critical points for $F(\varepsilon,\cdot)$ in N, for ε small. By Lemma 3 these critical points give rise to T-periodic solutions of (5) near Z, and Theorem 1 follows.

It is worth noticing that results such as Theorem 1 cannot be handled, in general, by the so-called «continuation methods» because here the linearized equation has 1 as a *double* eigenvalue. Even when a specific feature of the equation (as, for example, in the restricted 3-body problem) makes such a procedure applicable, it will give rise to the existence of solutions with period $T = T(\varepsilon)$ depending on ε. In particular, these results do not apply in our setting, which is concerned with time dependent perturbation. Ours is rather a bifurcation results. In fact the periodic solutions found above branch off from the «trivial» solutions Z, the bifurcation parameter being the coordinate in Z.

3. AN APPLICATION.

As a possible application of theorem 1 we consider the case in which $n = 2m$, $\phi(|q|) = 1/|q|^\alpha$ and $A(p,q) = (-\gamma J_m q, \gamma J_m p)$, where $\gamma \in \boldsymbol{R}$ and J_m denotes the symplectix matrix in $\boldsymbol{R}^{2m}$. Systems of this kind typically arise when one deals with motions referred to a frame rotating with angular velocity γ.

Using complex coordinates the unperturbed system (6), with the choice above, becomes

$$\frac{d^2 q}{dt^2} + 2i\gamma \frac{dq}{dt} - \gamma^2 q + \alpha \frac{q}{|q|^{\alpha+2}} = 0 \tag{7}$$

and has circular solutions of the form

$$Z = \left\{ q = R\xi \exp[i\omega t], \xi \in \mathbb{C}^m, |\xi| = 1, \alpha \frac{1}{R^{\alpha+1}} = (\omega + \gamma)^2, R > 0, \omega \neq 0 \right\}. \tag{8}$$

In order that Z verifies (Z) a non-resonance condition involving γ, ω and α is required. Precisely one shows:

LEMMA 5. *Set* $\lambda = \gamma/\omega$ *and suppose that*

$$(1+\lambda)\sqrt{2-\alpha} \notin \mathbb{Z}, \qquad \forall \alpha < 2, \textit{ and } 2\lambda \notin \mathbb{Z}. \tag{9}$$

Then Z given by (8) satisfies (Z).

Taking into account Lemma 5, an application of Theorem 1 yields:

THEOREM 6. *Let us consider the Hamiltonian System* (5) *with H_ε satisfying* (4) *where ϕ and A are as above. Moreover let Z be given by* (8) *and suppose that* (9) *holds Then* (5) *has, for ε small, at least two T-periodic solutions near Z.*

In the case of the Kepler's potential $\phi(|q|) = 1/|q|$ which includes, as a particular example, time dependent perturbations of the restricted, planar 3-body problem, the non-resonance condition (9) becomes simply $2\lambda \notin \mathbb{Z}$. Let us point out that in such a case Theorem 6 applies to any W, in contrast with the results of [2].

In fact, in [2], taking $V = -1/|q| + \varepsilon U(t,q)$, the perturbation $U(t,q)$ is required to satisfy:

$$U(t,-q) = U(t,q) \quad \text{and} \quad U\left(t + \frac{T}{2}, q\right) = U(t,q);$$

on the contrary, here, no restriction has to be imposed on the perturbation term W. The branching off from the circular solutions Z can be related to a sort of symmetry breaking, which in the present case is due not only to W but also do Coriolis forces.

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