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Perturbation results for a class of singular Hamiltonian systems

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<http://www.bdim.eu/item?id=RLINA_1989_8_83_1_129_0>
Equazioni differenziali. — Perturbation results for a class of singular Hamiltonian systems. Nota di Antonio Ambrosetti (*) e Ivar Ekeland (**), presentata (**) dal Corrisp. A. Ambrosetti.

ABSTRACT. — The existence of solutions with prescribed period $T$ for a class of Hamiltonian systems with a Keplerian singularity is discussed.

KEY WORDS: Hamiltonian systems; Singular nonlinearities; Critical point theory.

RIASSUNTO. — Risultati di perturbazione per una classe di sistemi Hamiltoniani con singolarità. Viene discussa l’esistenza di soluzioni di periodo assegnato $T$ per una classe di sistemi Hamiltoniani con singolarità di tipo Kepleriano.

1. INTRODUCTION.

This note deals with the existence of $T$-periodic solutions of a class of Hamiltonian systems with $n$ degrees of freedom

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p},$$

where the Hamiltonian $H = H(t, p, q)$ is $T$-periodic in $t$ and has a singularity at $q = 0$.

In some recent papers the case in which $H = (1/2)|p|^2 + V(t, q)$ and $V(t, \cdot)$ behaves like $-1/|q|^a$ near $q = 0$ has been investigated. More precisely, in [1] the existence of infinitely many $T$-periodic solutions not crossing the singularity $q = 0$ (non-collision orbits) is proved, under the main restriction that $a \geq 2$. On the other hand, in [2] $V$ has been taken of the more particular form: $V = -1/|q|^a + cU(t, q)$. In the last case any $a > 0$ is allowed (when $a = 1$ $U$ is supposed to even in $q$ and $(T/2)$-periodic in $t$) and it is shown that $T$-periodic solutions «branch off» from the circular orbits of the unperturbed system $q'' + a(q/|q|^{a+2}) = 0$.

In all the preceding papers (1) reduces to the second order system

$$\frac{d^2 q}{dt^2} + \frac{\partial V}{\partial q} = 0,$$

whose non-collision orbits are the critical points of the functional

$$f: H^1(S^1; R^n - \{0\}) \to R, \quad f(q) = \int_0^T \left\{\frac{1}{2}|q|^2 - V(t, q)\right\} dt,$$

which is suitable either for the Morse theory or for some perturbation results in critical point theory [3]. For other results concerning second order systems with singular potentials we refer, for example to [5, 6, 7].

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(***) Nella seduta dell’11 febbraio 1989.

In the present note we outline the arguments which permit the extension of the results of [2] to a class of first order Hamiltonian System such as (1). A more complete discussion as well as detailed proofs are contained in a forthcoming paper [4].

2. The main existence results.

Set $\Omega = \mathbb{R}^n \times \mathbb{R}^n - (\mathbb{R}^n \times \{0\})$ and let $K_e, H_e \in C^2(\mathbb{R} \times \Omega; \mathbb{R})$ be given by

\begin{align}
K_e(t, p, q) &= \frac{1}{2} |p|^2 - \phi(|q|) + \varepsilon W(t, p, q), \\
H_e(t, p, q) &= \frac{1}{2} (Az, z) + K_e(t, p, q),
\end{align}

where $z = (p, q) \in \Omega$, $(\cdot, \cdot)$ denotes the Euclidean scalar product, $A$ is a symmetric matrix in $\mathbb{R}^{2n}$, $W$ is (smooth and) $T$-periodic in $t$ and $\phi \in C^1(\mathbb{R}^n; \mathbb{R}^+)$ is convex and $\phi(r) \to +\infty$ as $r \to 0+$.

We denote by $J$ the symplectic matrix $J(p, q) = (q, -p)$ and look for $T$-periodic solutions of the Hamiltonian System

\begin{equation}
-\frac{d}{dt} = \nabla H_e(t, z).
\end{equation}

near those of the unperturbed autonomous system

\begin{equation}
-\frac{d}{dt} = \nabla H_0(z).
\end{equation}

Let $E = C(R/TZ; \mathbb{R}^{2n})$, $Z$ denote a set of $T$-periodic solutions of (6) such that $\Omega \ni B = B_2 = : \{z(t)\mid t \in Z, z \in Z\}$. For $z \in Z$ we denote by $Y(z)$ the set of the $T$-periodic solutions of

\begin{equation}
-\frac{d}{dt} y = H_0(z(t)) y,
\end{equation}

and suppose that:

\begin{itemize}
  \item $(Z)$ $Z$ is a $C^1$-submanifold of $E$, and for the tangent space $T_zZ$ there results: $Y(Z) = T_zZ \forall z \in Z$.
  \item Let $\text{cat} (\cdot)$ denote the Lusternik-Schnirelman category. Our main existence result is:
\end{itemize}

**Theorem 1.** Let us suppose that (4) and $(Z)$ hold. Moreover we assume that

(A) the matrix $A$ has no eigenvalue of the form $i k \omega$, $\omega = 2\pi/T$, $k \in \mathbb{Z}$.

Then there exist a neighbourhood $N'$ of $Z$ and $\varepsilon' > 0$ such that $V(|\varepsilon| < \varepsilon'$ (5) has at least $\text{cat}(Z)$ $T$-periodic solutions in $N'$.

The proof of Theorem 1 is based on a local variational principle suitable for the application of the perturbation results of [3]. For this let us start with some preliminary lemmas.

Let $B^* = \nabla K_0(B) = \{z^* = (p^*, q^*) \in \Omega\mid p^* = p, q^* = -\phi'(|q|)(q/|q|), (p, q) \in B\}$ and denote by $\phi^*$ the Legendre transform of $\phi$. It is possible to prove:

**Lemma 2.** There exist $\varepsilon^* > 0$, a neighbourhood $U$ of $B^*$ in $\Omega$ and $W^* \in C^2(\{|\varepsilon| < \varepsilon^*\} \times \mathbb{R} \times U; \mathbb{R})$ such that, letting $K(\varepsilon, t, z^*) = (1/2)|p^*|^2 - \phi^*(|q^*|) - W^*(\varepsilon, t, z^*)$, there re-
\[ K^*(0, z^0) = \frac{1}{2} |p^*|^2 - \phi^*(|q^*|); \]
\[ \nabla K^*(t, z^0) = z \quad \text{iff} \quad z \in \Omega \quad \text{and} \quad \nabla K_0(t, z) = z^0. \]

Next, we note that assumption (A) allows us to define a linear, compact, selfadjoint operator \( L: E \to E \) by setting
\[ z = Lw \quad \text{iff} \quad -\int \frac{dz}{dt} - Az = v. \]

Let \( N = \{ u \in E | u(t) \in U \} \) and define \( F \in C^2(\{ |\epsilon| < \epsilon^* \} \times N; \mathbb{R}) \) by setting
\[ F(\epsilon, u) = \int_0^T \left[ \frac{1}{2} (Lw, w) - K^*(\epsilon, t, u) \right] dt. \]

We are now in position to state our Local Variational Principle:

**Lemma 3.** If \( u \in N \) is a critical point of \( F(\epsilon, \cdot) \) then \( z = \nabla K^*(\epsilon, t, u) \in B \) is a \( T \)-periodic solution of (5).

Note that the Local Variational Principle stated above fits into the framework of the so-called Dual Action Principle. In fact, the Clarke's Dual Principle cannot be applied directly here because \( H_\epsilon \) is singular and convex in \( p \) but concave in \( q \).

**Lemma 3** leads to seek for critical points of \( F \) in \( N \). This will be done by using the perturbation theory developed in [3]. Set \( Z^* = \nabla K_0(Z) \) and \( F = F(0, \cdot) \).

**Lemma 4.** If \( (Z) \) holds then \( Z^* \) is non-degenerate critical manifold for \( F \), in the sense that \( \nabla F(u) = 0 \) and \( T_* Z^* = \text{Ker} \{ F''(u) \} \) for all \( u \in Z^* \).

Lemma 4 enables us to apply the results of [3] which yield the existence of \( \text{cat} (Z^*) \) (= \( \text{cat} (Z) \)) critical points for \( F(\epsilon, \cdot) \) in \( N \), for \( \epsilon \) small. By Lemma 3 these critical points give rise to \( T \)-periodic solutions of (5) near \( Z \), and Theorem 1 follows.

It is worth noticing that results such as Theorem 1 cannot be handled, in general, by the so-called «continuation methods» because here the linearized equation has 1 as a double eigenvalue. Even when a specific feature of the equation (as, for example, in the restricted 3-body problem) makes such a procedure applicable, it will give rise to the existence of solutions with period \( T = T(\epsilon) \) depending on \( \epsilon \). In particular, these results do not apply in our setting, which is concerned with time dependent perturbation. Ours is rather a bifurcation results. In fact the periodic solutions found above branch off from the «trivial» solutions \( Z \), the bifurcation parameter being the coordinate in \( Z \).

### 3. AN APPLICATION.

As a possible application of theorem 1 we consider the case in which \( n = 2m \), \( \phi(|q|) = 1/|q|^s \) and \( A(p, q) = (-\gamma J_m q, \gamma J_m p) \), where \( \gamma \in \mathbb{R} \) and \( J_m \) denotes the symplectic matrix in \( \mathbb{R}^{2m} \). Systems of this kind typically arise when one deals with motions referred to a frame rotating with angular velocity \( \gamma \).

Using complex coordinates the unperturbed system (6), with the choice above,
becomes

\[
\frac{d^2 \varphi}{dt^2} + 2i\gamma \frac{d\varphi}{dt} - \gamma^2 q + \varphi - \frac{q}{|q|^{n+2}} = 0
\]

and has circular solutions of the form

\[
Z = \left\{ q = R^\xi \exp \{ i\omega t \}, \xi \in \mathbb{C}^n, |\xi| = 1, \frac{1}{R^{n+1}} = (\omega + \gamma)^2, R > 0, \omega \neq 0 \right\}.
\]

In order that \( Z \) verifies (Z) a non-resonance condition involving \( \gamma, \omega \) and \( a \) is required. Precisely one shows:

**Lemma 5.** Set \( \lambda = \gamma/\omega \) and suppose that

\[
(1 + \lambda) \sqrt{2 - \alpha} \notin Z, \quad \forall \alpha < 2, \text{ and } 2\lambda \notin Z.
\]

Then \( Z \) given by \( (8) \) satisfies \( (Z) \).

Taking into account Lemma 5, an application of Theorem 1 yields:

**Theorem 6.** Let us consider the Hamiltonian System (5) with \( H \) satisfying (4) where \( \phi \) and \( A \) are as above. Moreover let \( Z \) be given by \( (8) \) and suppose that \( (9) \) holds. Then (5) has, for \( \varepsilon \) small, at least two \( T \)-periodic solutions near \( Z \).

In the case of the Kepler’s potential \( \phi(|\varphi|) = 1/|\varphi| \) which includes, as a particular example, time dependent perturbations of the restricted, planar 3-body problem, the non-resonance condition (9) becomes simply \( 2\lambda \notin Z \). Let us point out that in such a case Theorem 6 applies to any \( W \), in contrast with the results of [2].

In fact, in [2], taking \( V = -1/|\varphi| + \varepsilon U(t, \varphi) \), the perturbation \( U(t, \varphi) \) is required to satisfy:

\[
U(t, -\varphi) = U(t, \varphi) \quad \text{and} \quad U\left(t + \frac{T}{2}, \varphi\right) = U(t, \varphi);
\]
on the contrary, here, no restriction has to be imposed on the perturbation term \( W \). The branching off from the circular solutions \( Z \) can be related to a sort of symmetry breaking, which in the present case is due not only to \( W \) but also do Coriolis forces.

**References**


