Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Jerry Bartolomeo, Irena Lasiecka, Roberto Triggiani

Uniform exponential energy decay of Euler-Bernoulli equations by suitable boundary feedback operators

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **83** (1989), n.1, p. 121–128.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1989_8_83_1_121_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1989.

Atti Acc. Lincei Rend. fis.
(8), LXXXIII (1989), pp. 121-128

Equazioni a derivate parziali. — *Uniform exponential energy decay of Euler-Bernoulli equations by suitable boundary feedback operators* (*). Nota (**) di Jerry Bartolomeo, Irena Lasiecka e Roberto Triggiani, presentata dal Corrisp. R. Conti.

Abstract. — We study the uniform stabilization problem for the Euler-Bernoulli equation defined on a smooth bounded domain of any dimension with feedback dissipative operators in various boundary conditions.

Key words: Euler-Bernoulli equations; Uniform stabilization.

Riassunto. — *Decadimento uniforme esponenziale dell'energia nelle equazioni di Euler-Bernoulli con opportuna dissipazione sulla frontiera.* Studiamo, al variare delle condizioni al contorno, il problema di stabilizzazione uniforme per l'equazione di Euler-Bernoulli con dissipazione definita su un dominio regolare limitato di dimensione qualunque.

0. Introduction.

Throughout this note Ω is an open bounded domain in R^n with sufficiently smooth boundary $\partial\Omega = \Gamma$. In Ω we consider the Euler-Bernoulli equation with suitable boundary conditions which, once homogeneous (free dynamics), produce a *unitary* group of operators, i.e. *norm-preserving* free solutions on suitable natural function spaces. We then seek to introduce «damping» in the dynamics by virtue of expressing the non-homogeneous boundary controls as suitable feedback operators in terms of the velocity, in order to force *uniform exponential decay* of all feedback solutions. In section 1, we treat boundary controls in the Direchlet and Neumann boundary conditions. Here, the uniform stabilization results which we present are fully consistent with recently established exact controllability and optimal regularity theories [11, 12, 14, 15], (which, in fact, motivate the choices of spaces in the first place). In section 2, a bending moment type of condition replaces the Neumann boundary condition. Here further difficulties arise. We present results when only one boundary control is active. Thus, as expected, geometrical conditions on Ω are needed. These, however, are more restrictive than in section 1. For work in the stabilization of plates with other boundary conditions, we refer to [5-9, 13].

1. Euler-Bernoulli equation with Dirichlet and Neumann boundary controls [1].

Consider the mixed problem

$$w_{tt} + \Delta^2 w = 0 \qquad \text{in } (0, T] \times \Omega = Q, \tag{1.1a}$$

$$w(0, \cdot) = w_0; \quad w_t(0, \cdot) = w_1 \qquad \text{in } \Omega, \tag{1.1b}$$

(*) Research was sponsored partially by the National Science Foundation Grant No.
(**) Pervenuta all'Accademia il 10 ottobre 1988.

$$w|_{\Sigma} = g_1 \quad \text{in } (0, T] \times \Gamma = \Sigma, \tag{1.1c}$$

$$\left.\frac{\partial w}{\partial \nu}\right|_{\Sigma} = g_2 \quad \text{in } \Sigma, \tag{1.1d}$$

in the solution $w(t,x)$ subject to suitable control functions g_1 and g_2 [14, 15]. Throughout this section let A be the positive self-adjoint operator on $L^2(\Omega)$ defined by $Af = \Delta^2 f$, $\mathcal{D}(A) = H^4(\Omega) \cap H_0^2(\Omega)$. We introduce the following spaces, as in recent exact controllability studies [14, 15]

$$X = H^{-1}(\Omega) \times V' \tag{1.2}$$

$$V = \left\{ f \in H^3(\Omega) : f|_{\Gamma} = \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma} = 0 \right\} \tag{1.3}$$

$$Y = H_0^1(\Omega) \times H^{-1}(\Omega) \tag{1.4}$$

where we recall that

$$\mathcal{D}(A^{1/4}) = H_0^1(\Omega); \qquad \mathcal{D}(A^{3/4}) = V \tag{1.5}$$

(with equivalent norms), so that the spaces X and Y in (1.2), (1.4) can likewise be identified as

$$X = [\mathcal{D}(A^{1/4})]' \times [\mathcal{D}(A^{3/4})]' \tag{1.6}$$

$$Y = \mathcal{D}(A^{1/4}) \times [\mathcal{D}(A^{1/4})]'. \tag{1.7}$$

The symbol $'$ denotes duality with respect to the $L^2(\Omega)$-topology. The norms are given by

$$\|x\|_{\mathcal{D}(A^{\alpha})} = \|A^{\alpha} x\|_{L^2(\Omega)}; \qquad \|x\|_{[\mathcal{D}(A^{\beta})]'} = \|A^{-\beta} x\|_{L^2(\Omega)} \tag{1.8}$$

where α, $\beta \geq 0$. The problem of *exact controllability* for the dynamics (1.1) on the spaces X and Y with either one or else two controls g_1 and g_2 in suitable functions spaces was studied in [14, 15]. To help motivate the spaces of uniform stabilization chosen below, we recall from these references that *e.g.* exact controllability of (1.1) in the space X of *optimal* regularity was obtained for an arbitrarily short time $T > 0$ either with control functions

$$g_1 \in L^2(0, T; L^2(\Gamma)); \qquad g_2 \equiv 0 \tag{1.9}$$

under geometrical conditions for Ω; or else with controls

$$g_1 \in L^2(0, T; L^2(\Gamma)); \qquad g_2 \in L^2(0, T; H^{-1}(\Gamma)) \tag{1.10}$$

without geometrical conditions on Ω (except for smoothness of Γ). Thus the corresponding *uniform stabilization* problem may be stated qualitatively as follows: seek, if possible, two (linear) feedback operators $\mathcal{F}_1$ and $\mathcal{F}_2$

$$g_1 = \mathcal{F}_1(w_t) \in L^2(0, \infty; L^2(\Gamma)) \tag{1.11a}$$

$$g_2 = \mathcal{F}_2(w_t) \in L^2(0, \infty; H^{-1}(\Gamma)) \tag{1.11b}$$

based on the velocity w_t (damping) such that the corresponding closed loop problem which results from introducing (1.11*a-b*) into (1.1*c-d*), respectively, is (i) well-posed on X (in the sense of semigroup generation on X) and (ii) decays (exponentially) in the uniform operator topology of X for $t \to +\infty$. Solution to this problem is provided by the following two theorems.

THEOREM 1.1 [1]. (Well-posedness and uniform stabilization on X with two feedback operators). Consider problem (1.1) with

$$g_1 = -\frac{\partial}{\partial \nu}(\Delta A^{-3/2} w_t)|_\Gamma \tag{1.12}$$

$$g_2 = \Lambda^2 [\Delta (A^{-3/2} w_t)]_\Gamma \tag{1.13}$$

where Λ denotes an isomorphism $H^s(\Gamma)$ onto $H^{s-1}(\Gamma)$ self-adjoint on $L^2(\Gamma)$ (we need only the case $s = 1$) (first order differential operator tangential to Γ with smooth coefficients: say, tangential gradient). Then, the closed loop problem obtained from inserting (1.12) and (1.13) into (1.1*c*) and (1.1*d*), respectively, possesses the following properties:

(i) (well-posedness) the solution map

$$\{w_0, w_1\} \to \{w(t), w_t(t)\} \tag{1.14}$$

defines a strongly continuous contraction semigroup $\exp[\mathcal{A}t]$ on X;

(ii) (L_2-nature in time of feedback operators). The functions g_1, g_2 given by (1.12) (1.13) satisfy the inequality

$$\int_0^\infty \{\|g_1(t)\|^2_{L^2(\Gamma)} + \|g_2(t)\|^2_{H^{-1}(\Gamma)}\}\, dt \leq \|\{w^0, w^1\}\|^2_X; \tag{1.15}$$

(iii) (uniform stabilization) there exist constants M and $\delta > 0$ such that

$$\left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right\|_X = \left\| \exp[\mathcal{A}t] \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_X \leq M \exp[-\delta t] \left\| \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_X, \qquad t \geq 0 \quad \square. \tag{1.16}$$

We note explicitly that no geometrical conditions are imposed on Ω in Theorem 1.1 (except for smoothness of Γ). If, however, only one feedback control action g_1 is used, then uniform stabilization is still achieved, but under geometrical conditions on Ω.

THEOREM 1.2 [1] (Uniform stabilization on X with one feedback operator). The same conclusions of Theorem 1.1 continue to hold true if g_1 is given by (1.12) while g_2 is taken identically zero $g_2 \equiv 0$, provided Ω satisfies the following geometrical condition: There exists a vector field $h(x) \in C^2(\overline{\Omega})$ such that

$$\text{(i)} \quad h \cdot \nu \geq \gamma > 0 \text{ on } \Gamma, \qquad \nu = \text{unit outward normal}, \qquad \gamma = \text{constant} \tag{1.17}$$

$$\text{(ii)} \quad \int_\Omega H(x)\, v(x) \cdot v(x)\, d\Omega \geq \rho \int_\Omega |v(x)|^2_{\mathbb{R}^n}\, d\Omega, \qquad \text{for some constant } \rho > 0 \tag{1.18}$$

and all $v(x) \in [L^2(\Omega)]^n$, where $H(x)$ is the $n \times n$ matrix with $\partial h_i(x)/\partial x_j$ as its (i,j)-th entry (H is the transpose of the Jacobian of h). We note that a sufficient checkable condition for (ii) to hold is that the symmetric matrix $H(x) + H^*(x)$ be uniformly positive definite on $\overline{\Omega}$. $\square$

Next, if one wishes to study uniform stabilization of (1.1) in the space Y given by (1.4) (and corresponding to the exact controllability results in this space of [14, 15]), one sees from (1.4) that we must take at the outset $g_1 \equiv 0$. A study of this case is presented in [1]. We close this section by remarking that the proof of Theorems 1.1 and 1.2 are inspired by the uniform stabilization papers [17] for the wave equation with Dirichlet feedback; [21] for the wave equation with Neumann feedback following the original contributions of G. Chen [2, 3] and J. Lagnese [10]; and [14, 15] for the corresponding exact controllability results for (1.1).

2. Euler-Bernoulli equation with boundary controls on $w|_\Sigma$ and $\Delta w|_\Sigma$ [20].

Throughout this section we consider the problem

$$w_{tt} + \Delta^2 w = 0 \qquad \text{in } (0,T] \times \Omega = Q, \tag{2.1a}$$

$$w(0,\cdot) = w_0; \quad w_t(0,\cdot) = w_1 \qquad \text{in } \Omega, \tag{2.1b}$$

$$w|_\Sigma = g_1 \qquad \text{in } (0,T] \times \Gamma = \Sigma, \tag{2.1c}$$

$$\Delta w|_\Sigma = g_2 \qquad \text{in } \Sigma, \tag{2.1d}$$

Regularity results (in fact, optimal) in appropriate function spaces were given in [18], while corresponding controllability results may be found in [10, 19]. Here we study the uniform stabilization problem for (2.1) on the spaces

$$Z = \mathcal{D}(A^{1/2}) \times L^2(\Omega)\,. \tag{2.2}$$

$$W = L^2(\Omega) \times [\mathcal{D}(A^{1/2})]'\,. \tag{2.3}$$

In (2.2)-(2.3), and throughout this section, we let $Af = \Delta^2 f$, $\mathcal{D}(A) = \{f \in H^4(\Omega) : f|_\Gamma = \Delta f|_\Gamma = 0\}$ and we have $A^{1/2} f = -\Delta f$, $\mathcal{D}(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega)$. Hence, choice (2.2) of Z implies at the outset the condition

$$g_1 \equiv 0 \qquad \text{on } \Sigma \tag{2.4}$$

on problem (2.1), and we are thus left only with the control g_2 (to be suitable expressed as a feedback operator on w_t as in (2.5) below) to force *uniform stabilization* for (2.1). Not surprisingly, geometrical conditions on Ω are needed to achieve uniform stabilization. As a matter of fact, these are more restrictive, and of more difficult interpretation as well, on the class of domains Ω which are allowed. They are described in Definition 2.1 which follows the statement of Theorem 2.1.

Theorem 2.1 [20] (Well-posedness and uniform stabilization on Z). Consider problem (2.1) with

$$g_1 \equiv 0, \quad g_2 = -\left.\frac{\partial w_t}{\partial \nu}\right|_\Sigma . \tag{2.5}$$

Then:

(i) (well-posedness) the corresponding map $\{w_0, w_1\} \to \{w(t), w_t(t)\}$ defines a strongly continuous contraction semigroup $\exp[\mathcal{A}t]$ on Z;

(ii) (L_2-nature of feedback operators) The feedback control g_2 in (2.5) satisfies the inequality

$$\int_0^\infty \|g_2(t)\|^2_{L^2(\Gamma)}\, dt \leq \|\{w_0, w_1\}\|^2_Z .$$

(iii) (Uniform stabilization on Z) Let now Ω satisfy the geometrical conditions of definition 2.1 below.

Then there exist constants M and $\delta > 0$ such that

$$\left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right\|_Z = \left\| \exp[\mathcal{A}t] \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z \leq M \exp[-\delta t] \left\| \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z , \qquad t \geq 0 \quad \square . \tag{2.6}$$

A similar result holds true on the space W defined by (2.3), mutatis mutandis, see [20].

The proof of Theorem 2.1 is inspired by paper [17] on the uniform stabilization of the wave equation with Dirichlet feedback; in particular, it presents a technical difficulty of the same type as one encountered in [17].

DEFINITION 2.1. Let Ω satisfy the following condition. There exists a vector field $h(x) \in C^2(\overline{\Omega})$ such that:

(i) h is parallel to ν (exterior unit normal) on all of Γ; *i.e.* $h(\sigma) = k(\sigma)\,\nu(\sigma)$, for $k(\sigma)$ a smooth scalar function, $\sigma \in \Gamma$;

(ii) the following inequality holds

$$\int_\Omega \Delta q \left[\sum_{i=1}^n \nabla h_i \cdot \nabla q_{x_i} \right] d\Omega \geq \rho \int_\Omega |\Delta q|^2\, d\Omega$$

where $q(x)$ is a smooth function on Ω such that

$$q|_\Gamma \equiv 0 \quad \text{and} \quad \Delta q|_\Gamma \equiv 0$$

and $\rho > 0$ is a suitable constant, possible depending on $h(x)$, Ω, and $q(x)$. $\square$

Examples of domains satisfying Definition 2.1 include n-dimensional spheres with centre x_0, where $h(x) = x - x_0$ and n-dimensional ellipsoids where the ratio between the axes is «sufficiently small».

3. SKETCH OF PROOF OF THEOREMS 1.1 AND 1.2.

As the proof of Theorems 1.1 and 1.2 is lengthy, we can only confine ourselves here to a schematic sketch which will be concentrated on the main issue of uniform stabilization. (Well-posedness in the semigroup sense and L_2-nature of the feedback operators follow from a dissipative type of argument based on Lumer-Phillips theorem as in [17,21]). We define (recall (1.2), (1.6), (1.8))

$$E(w,t) \equiv E(t) = \|\{w(t), w_t(t)\}\|^2_X = \|A^{-1/4} w(t)\|^2_{L^2(\Omega)} + \|A^{-3/4} w_t(t)\|^2_{L^2(\Omega)} \leq E(0) . \tag{3.1}$$

Our main goal will be, as usual, to show that

$$\int_0^\infty E(t)\, dt \leq \text{const}\, E(0) \qquad \forall \{w_0, w_1\} \in X \tag{3.2}$$

with «const» independent of the initial data. After this, Datko's Theorem will yield the desired conclusion. As a matter of fact, it will suffice to show inequality (3.2) for initial data smooth, say in the domain of the generator of the feedback problem. To this end, we first introduce a new variable p by setting

$$p = A^{-3/2} w_t \tag{3.3}$$

which yields the new problem

$$p_{tt} + \Delta^2 p = F_1 + F_2 \qquad \text{in } (0,\infty)\times\Omega = Q, \tag{3.4a}$$

$$p_0 = A^{-3/2} w_1; \quad p_1 = A^{-3/2} w_{tt}(0) \qquad \text{in } \Omega, \tag{3.4b}$$

$$p|_\Sigma \equiv 0 \quad \text{and} \quad \left.\frac{\partial p}{\partial \nu}\right|_\Sigma \equiv 0 \qquad \text{in } (0,\infty)\times\Gamma = \Sigma, \tag{3.4c}$$

$$F_1 = -A^{-1/2} G_1 G_1^* A^{-1/2} w_{tt}; \quad F_2 = -A^{-1/2} G_2 \Lambda^2 G_2^* A^{-1/2} w_{tt} \tag{3.5}$$

$$G_1 g_1 = v \Leftrightarrow \left\{ \Delta^2 v = 0 \text{ in } \Omega;\ v|_\Gamma = g_1;\ \left.\frac{\partial v}{\partial \nu}\right|_\Gamma = 0 \right\} \tag{3.6}$$

$$G_2 g_2 = y \Leftrightarrow \left\{ \Delta^2 y = 0 \text{ in } \Omega;\ y|_\Gamma = 0;\ \left.\frac{\partial y}{\partial \nu}\right|_\Gamma = g_2 \right\} \tag{3.7}$$

By (3.3) and the w-problem it follows that

$$\|A^{-3/4} w_t\|_{L^2(\Omega)} = \|A^{3/4} p\|_{L^2(\Omega)}, \qquad \text{equivalent to} \quad \left\{ \int_\Omega |\nabla(\Delta p)|^2 \, d\Omega \right\}^{1/2} \tag{3.8}$$

$$A^{1/4} p_t = -A^{-1/4} w + \mathcal{O}(\|g_1\|_{L^2(\Gamma)}) + \mathcal{O}(\|A^{-1} g_2\|_{L^2(\Gamma)}) \tag{3.9}$$

$$\|A^{1/4} p_t\|_{L^2(\Omega)}, \qquad \text{equivalent to} \quad \left\{ \int_\Omega |\nabla p_t|^2 \, d\Omega \right\}^{1/2} \tag{3.10}$$

with g_1 and g_2 in feedback form as in (1.12), (1.13), which therefore satisfy (1.15). The equivalences in (3.8) and (3.10) have been pointed out and crucially used in the exact controllability study in [14, 15], (after Grisvard's interpolation results). Next, following an idea in the exact controllability study in [14, 15], we apply to the p-problem (3.4) two multipliers: $\exp[-2\beta t]\, h \cdot \nabla(\Delta p)$ and $\exp[-2\beta t]\, \nabla p \operatorname{div} h$, $\beta > 0$, with h the vector field for Ω (this is the vector field h of the statement of Theorem 1.2 if only the feedback on g_1 is used, while $g_2 \equiv 0$; instead, it may be a radial vector field $x - x_0$ in the case of both feedbacks on g_1 and g_2). After (lengthy) integrations by parts one obtains the following identity after using the boundary conditions

$$\begin{aligned} &\int_\Sigma \exp[-2\beta t] \frac{\partial(\Delta p)}{\partial \nu} h \cdot \nabla(\Delta p)\, d\Sigma - 1/2 \int_\Sigma \exp[-2\beta t] |\nabla(\Delta p)|^2 h \cdot \nu \, d\Sigma + \\ &\qquad + 1/2 \int_\Sigma \exp[-2\beta t]\, \Delta p \frac{\partial(\Delta p)}{\partial \nu} \operatorname{div} h \, d\Sigma = \\ &= \int_Q \exp[-2\beta t]\, H \nabla(\Delta p) \cdot \nabla(\Delta p)\, dQ + \int_Q \exp[-2\beta t]\, H \nabla p_t \cdot \nabla p_t \, dQ + \end{aligned} \tag{3.11}$$

$$+ 1/2 \int_Q \exp[-2\beta t]\, \Delta p\, \nabla(\Delta p)\cdot\nabla(\operatorname{div} h)\, dQ + 1/2 \int_Q \exp[-2\beta t]\, p_t \nabla p_t \cdot \nabla(\operatorname{div} h)\, dQ -$$

$$- \beta \int_Q \exp[-2\beta t]\, p_t \Delta p \operatorname{div} h\, dQ - 2\beta \int_Q \exp[-2\beta t]\, p_t h\cdot\nabla(\Delta p)\, dQ +$$

$$+ 1/2 \int_Q \exp[-2\beta t]\, F \Delta p \operatorname{div} h\, dQ + \int_Q \exp[-2\beta t]\, F h \cdot \nabla(\Delta p)\, dQ -$$

$$- 1/2 \int_\Omega \nabla p_0 \cdot \nabla(p_1 \operatorname{div} h)\, d\Omega + \int_\Omega p_1 h \cdot \nabla(\Delta p_0)\, d\Omega$$

where $F = F_1 + F_2$, as the terms of integration by parts in t as $T \to \infty$ vanish (we are taking smooth initial data, as mentioned below (3.2)) [Note that $\operatorname{div} h \equiv 0$ if h is a radial field, the case of two feedbacks]. Technical manipulations on the terms of (3.11) using the norm equivalences (3.8), (3.9), (3.10) for the interior terms, and (1.15) for the boundary terms yield the following estimates for the left hand side (L.H.S) and right hand side (R.H.S.) of identity (3.11):

$$C_h E(0) \geq C_h \left\{ \int_0^\infty [\|g_1\|^2_{L^2(\Gamma)} + \|g_2\|^2_{H^{-1}(\Gamma)}] \right\} dt \geq \text{L.H.S. of (3.11)} \tag{3.12}$$

$$\text{R.H.S. of (3.11)} \geq c_h^2 \int_0^\infty \exp[-2\beta t]\, E(t)\, dt - K_h^2 E(0) \tag{3.13}$$

in the case of radial vector field (Theorem 1.1), where the positive constants c_h^2 and K_h^2 do not depend on β. Combining (3.12), (3.13) and letting $\beta \downarrow 0$ yields (3.2) as desired. In the case of general vector field h the right hand side contains also an additional term: an integral over $\int_0^\infty$ in time with *lower order* terms in p (or w). These then can be «absorbed» through a theorem like Theorem 2 in [10] or Theorem 1.2 in [21] (our proof follows the operator proof in [21]. □

4. SKETCH OF PROOF OF THEOREM 2.1.

With reference to problem (2.1), (2.5), we now introduce a new variable p by setting $p = A^{-1/2} w_t$, where new A realizes Δ^2 with homogeneous boundary conditions $f|_\Gamma = \Delta f|_\Gamma = 0$. We thus obtain a corresponding problem in p:

$$p_{tt} + \Delta^2 p = F_1 \qquad p|_\Sigma = \Delta p|_\Sigma = 0 \qquad p_0, p_1 \tag{4.1}$$

$$F_1 = -G_1 G_1^* A^{1/2} w_{tt} \tag{4.2}$$

counterpart of (3.4), (3.5). Now however we apply the multipliers $\exp[-2\beta t]\, h\cdot\nabla p$ and $\exp[-2\beta t] \operatorname{div} h$ to the problem (4.1), thus obtaining a (lengthy) identity, counter part of identity (3.11). It is now the term in this identity due to $F_1 h \cdot \nabla p$ that gives rise to a technical difficulty of the same type as the one encountered in the wave equation with Dirichlet feedback in [17. Lemma 3.3]. This forces the condition that h be parallel to von Γ. Details are given in [20].

References

[1] J. Bartolomeo - R. Triggiani, 1988. *Uniform stabilization of the Euler-Bernoulli equation with Dirichlet and Neumann boundary feedback.* SIAM J. Mathematical Analysis, to appear.

[2] G. Chen, 1979. *Energy decay estimates and exact controllability of the wave equation in a bounded domain.* J. Math. Pures et Appl., (9), 58: 249-274.

[3] G. Chen, 1981. *A note on the boundary stabilization of the wave equation.* SIAM J. Control, 19: 106-113.

[4] F. Flandoli - I. Lasiecka - R. Triggiani, *Algebraic Riccati equations with non-smoothing observation arising in hyperbolic and Euler-Bernoulli equations.* Annali di Matematica Pura e Applicata, to appear.

[5] J. V. Kim - Y. Ranardy, 1987. *Boundary control of the Timoshenkso beam.* SIAM J. Control., 25: 1147-1429.

[6] J. Lagnese, 1988. Paper presented at International Workshop held in Vorau. Austria, July 10-16.

[7] J. Lagnese, 1989. *Boundary stabilization of thin elastic plates*, SIAM Studies in Applied Mathem.

[8] J. Lagnese, 1987. *Uniform boundary stabilization of homogeneous, isotropic plates.* Lectures Notes in Control Science, 102, Springer-Verlag: 204-215; Proceedings of the 1986 Vorau Conference on Distributed Parameter Systems.

[9] J. Lagnese, *A note on boundary stabilization of wave equations.* SIAM J. Control, to appear.

[10] J. Lagnese, 1983. *Decay of solutions of wave equations in a bounded region with boundary dissipation.* J. Diff. Eqts., 50, (2): 163-182.

[11] J. L. Lions, 1988. *Exact controllability, stabilization and perturbations.* SIAM Review, March 1988, to appear in extended version by Masson.

[12] J. L. Lions, 1986. *Un résultat de regularité* (paper dedicated to S. Mizohata), Current Topics on Partial Differential Equations, Kinokuniya Company, Tokyo.

[13] J. Lagnese - J. L. Lions, 1988. *Modelling, analysis and control of thin plates.* Masson.

[14] I. Lasiecka - R. Triggiani, 1987. *Exact controllability of the Euler-Bernoulli equation with $L^2(\Sigma)$-control only in the Dirichlet boundary conditions.* Atti della Accademia Nazionale dei Lincei, Rendiconti Classe di Scienze fisiche, matematiche e naturali, vol. 81, Roma.

[15] I. Lasiecka - R. Triggiani, 1989. *Exact controllability of the Euler-Bernoulli equation with controls in the Dirichlet and Neumann boundary conditions: a non-conservative case.* SIAM J. Control & Optimization, 27: 330-373.

[16] I. Lasiecka - R. Triggiani, 1987. *A direct approach to exact controllability for the wave equation with Neumann boundary control and to an Euler-Bernoulli equation.* Proceedings 26th IEEE Conference, 529-534, Los Angeles.

[17] I. Lasiecka - R. Triggiani, 1987. *Uniform exponential energy decay of the wave equation in a bounded region with $L_2(0,\infty;L_2(\Gamma))$-feedback control in the Dirichlet boundary conditions.* J. Diff. Eqs., 66: 340-390.

[18] I. Lasiecka - R. Triggiani, 1988. *Regularity theory for a class of non-homogeneous Euler Bernoulli equations: a cosine operator approach.* Bollettino Unione Matematica Italiana, (7), 3-B (1989): 199-228.

[19] I. Lasiecka - R. Triggiani, 1990. *Exact controllability of the Euler-Bernoulli equation with boundary controls for displacement and moments.* J. Mathem. Analysis & Applic., 146: 1-33.

[20] I. Lasiecka - R. Triggiani, 1989. *Uniform exponential energy decay of the Euler-Bernoulli equation on a bounded region with boundary feedback acting on the bending moment.* Dept. of Applied Mathem. Report, University of Virginia.

[21] R. Triggiani, 1987. *Wave equation on a bounded domain with boundary dissipation: an operator approach.* J. Mathem. Anal. & Applic., 37 (1989), 438-461; Operator Methods for Optimal Control Problems, Lectures Notes in Pure and Applied Mathematics, 108: 283-310 (. Lee Ed.), Marcel Dekker; also in Recent Advances in Communication and Control Theory, honoring the sixtieth anniversary of A. V. Balakrishnan (R. E. Kalman and G. I. Marchuk, Eds.), 262-286, Optimization Software (New York, 1987).

[22] I. Lasiecka - R. Triggiani, 1989. *Further results on exact controllability of the Euler-Bernoulli equation with controls in the Dirichlet and Neumann boundary conditions*, to appear.